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CONTINUOUS TIME VERSUS DISCRETE TIME IN THE NEW KEYNESIAN MODEL: CLOSED-FORM SOLUTIONS AND IMPLICATIONS FOR LIQUIDITY TRAP

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JEL Classification: C61, C63, C68, E31, E52

Keywords: forward guidance, continuous time, New Keynesian Model, ZLB. liquidity trap, closed-form solution

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Continuous Time Versus Discrete Time in the New Keynesian Model: Closed-Form Solutions and Implications for Liquidity Trap*

Lilia Maliar[†]

December 5, 2018

Abstract

Economists often use interchangeably the discrete- and continuous-time versions of the Keynesian model. In the paper, I ask whether or not the two versions effectively lead to the same implications. I analyze several alternative monetary policies, including a Taylor rule, discretionary interest rate choice and forward guidance. I show that the answer depends on a specific scenario and parameterization considered. In particular, in the presence of liquidity trap, the discrete-time analysis helps overcome some negative implications of the continuous-time model, such as excessively strong impact of price stickiness on inflation and output, unrealistically large government multipliers, as well as implausibly large effects of forward guidance.

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1 Introduction

The new Keynesian model is the workhorse of modern monetary policy analysis. Economists often use discrete-time and continuous-time models interchangeably. An important question is whether or not the two versions of the new Keynesian model lead to the same implications. I address this question in the present paper by comparing the properties of equilibria in the discrete- and continuous-time new Keynesian models under two alternative monetary polities. One is the Taylor-style rules in the absence of liquidity trap (an active zero lower bound, ZLB, on the nominal interest rate); and the other is a discretionary interest-rate setting in the presence of liquidity trap. I also analyze how such new Keynesian economies are affected by forward guidance (FG) – an unconventional monetary-policy tool of the central bank that consists in making announcements about future policy changes.

During liquidity-trap periods, the monetary authority cannot follow a Taylor rule. In an important contribution, Cochrane (2017) shows that if the monetary authority directly decides on the interest rate path in a discretionary fashion, then the resulting new Keynesian model has indeterminacy of equilibrium and features several important shortcomings: First, the model predicts that government spending can have huge output multipliers, even though such spending is entirely useless, for example, a purposeful destruction of capital or a technological regress can raise output. Second, if government adopts a policy directed at making prices more flexible, output multipliers become even larger, in particular, more flexible prices counterintuitively lead to worse deflation and depression in the liquidity-trap scenario. Finally, a FG policy, that consists in postponing interest-rate rises after the liquidity-trap period, can have enormous stimulative effects on output – the so-called FG puzzle; see Del Negro et al. (2015), McKay et al. (2016) and Maliar and Taylor (2018) for a discussion.[1]

I show that some of the above shortcomings of the stylized new Keynesian model can be either corrected or significantly ameliorated if one considers a discrete-time version of the model instead of the continuous-time version studied in Cochrane (2017). The difference between the continuous- and discrete-time models consists in that the former model has current inflation in the IS curve, while the latter model has future (next-period) inflation. As a result, the discrete-time model has two channels through which inflation affects output, one is inflation and the other is an expected change of inflation; in contrast, the continuous-time model has only the former channel. Forward in time, both inflation and inflation expectation channels act in the same direction and reinforce one another, however, backward in time, they act in opposite directions and partially offset one another. Precisely this mechanism helps correcting undesirable implication of the model highlighted in the continuous-time literature, in particular, unrealistically large government multipliers, an excessively strong impact of price stickiness on inflation and output and an implausibly large effects of FG.

[1] Werning (2012) studies the same model with a liquidity trap but focuses on characterizing the optimal monetary policy choice.

The expectation term also helps to discriminate among multiple equilibria. Cochrane (2017) suggests several criteria that can be used for the equilibrium refinement, such as (i) empirical plausibility – how well the model fits the data; (ii) the size of the effects – small frictions must imply small effects; (iii) boundedness of initial jumps; (iv) backward stability. Generally, equilibria satisfy one criteria but fail the others. For example, an equilibrium that enforces zero inflation at the end of the liquidity-trap period has a meaningful prediction that a liquidity trap leads to a recession with deflation (thus, it passes the test of empirical plausibility) but the magnitude of the recession is unrealistically large (thus, it fails to pass the test about the size of the effect). I show that the current- and future-inflation models have different implications on how well multiple equilibria satisfy the above equilibrium-selection criteria. For example, the equilibrium, in which inflation is set to zero at the end of the liquidity-trap period, no longer predicts unrealistically large effects on inflation and output in the future-inflation model (while the directions of changes are still correct).

The present paper has also two methodological contributions. First, for the continuous-time model, I establish the parameters ranges that lead to different types of characteristic roots (distinct real, repeated real or complex), construct closed-form solutions for all types of roots and explicitly characterize the corresponding impulse-responses functions. In discrete time, similar results are derived in Taylor and Maliar (2018); see also Taylor (1986) and Cochrane (2011) for related analysis. In continuous time, closed-form solutions are constructed only for the economy with liquidity trap in which there are two distinct real roots with opposite sign; see Cochrane (2017). My analysis fill in the gap: the constructed closed-form solutions include both the Taylor rule and discretionary interest-rate choice as two special cases. To the best of my knowledge, closed-form solutions for the cases of repeated real and complex roots have not been derived in the literature yet.

Second, I propose a convenient indirect way of comparing discrete- and continuous-time models when the direct comparison is complicated or infeasible. For example, some equilibria are constructed in the continuous-time literature by assuming that the economy arrives to zero inflation at the end of the FG period, which may result in FG duration of $\tau = 0.13$, for example. In the discrete-time economy, all events occurs on the set of natural numbers, $\tau = 0, 1, 2, ...$, so that there is no one-to-one correspondence between the continuous- and discrete-time models in the dimension of time. I show how to modify the continuous-time model so that it generates approximately the same dynamics as the discrete-time model. Instead of comparing the discrete- and continuous-time models, I compare the original and modified continuous-time models.

The rest of the paper is as follows: In Section 2, I modify the stylized continuous-time model to include next-period inflation in the IS curve. In Section 3, I establish the properties of the model with the Taylor rule and derive closed-form solutions. In Section 4, I use the constructed model to re-assess the properties of equilibrium and the effect of FG and government multipliers in the economy with liquidity trap; and finally, in Section 5, I conclude.

2 Continuous-time versus discrete-time new Keynesian models

I consider a stylized new Keynesian model in continuous time which consists of, respectively, an IS equation and Phillips curve expressed in deviations from the steady state

$$\frac{dx_t}{dt} = \sigma\left(i_t - i_t^n - \pi_t\right), \tag{1}$$

$$\frac{d\pi_t}{dt} = \rho\pi_t - k\left(x_t + g_t\right), \tag{2}$$

where x_t is the output gap; π_t is inflation; i_t is the nominal interest rate; i_t^n is the natural rate of interest; g_t is a government disturbance; k is a slope of the Phillips curve; σ is a coefficient of risk aversion; and ρ is the discount rate. The model (1), (2) is studied in, e.g., Werning (2012) and Cochrane (2017).

Next, I consider a deterministic stylized new Keynesian model in discrete time studied in, e.g., Woodford (2003) and Galí (2008),

$$x_t = x_{t+1} - \sigma\left(i_t - i_t^n - \pi_{t+1}\right), \tag{3}$$

$$\pi_t = \beta\pi_{t+1} + \kappa\left(x_t + g_t\right), \tag{4}$$

where β is the discount factor, and κ is the slope of the Phillips curve.

I observe that the continuous-time model contains current inflation π_t in the IS curve (2), while the discrete-time model contains next-period inflation π_{t+1} in the IS curve (3). How does this difference affect the implications of the new-Keynesian model?

To assess the differences between discrete- and continuous-time models, we can construct and compare the solutions to the two models directly. However, some interesting experiments are hard to implement in a comparable way in the two models, in particular, the experiments of Cochrane (2017) about liquidity trap. For example, one equilibrium constructed in Cochrane (2017) requires computing the length of FG period that makes the economy to reach zero inflation exactly at the end of the liquidity trap period: the resulting lengths are fractional real numbers, like $\tau = 0.13$ or $\tau = 0.6$. There is no direct counterpart of such an equilibrium in discrete time model, in which we have $\tau = 0, 1$.

Therefore, I use another indirect way of comparison of the two models, namely, I derive a continuous-time model that replicates dynamics of the discrete-time model. To this purpose, I approximate the differences in the discrete-time model (3), (4) with the derivatives, i.e., $\frac{x_{t+1}-x_t}{(t+1)-t} \approx \frac{dx_t}{dt}$ and $\frac{\pi_{t+1}-\pi_t}{(t+1)-t} \approx \frac{d\pi_t}{dt}$ which yields:

$$\frac{dx_t}{dt} = \sigma\left(i_t - i_t^n - \left(\pi_t + \frac{d\pi_t}{dt}\right)\right), \tag{5}$$

$$\frac{d\pi_t}{dt} = \rho\pi_t - k\left(x_t + g_t\right), \tag{6}$$

where $\rho \equiv \frac{1}{\beta} - 1$ and $k \equiv \frac{\kappa}{\beta}$ relate the parameters of the continuous- and discrete-time models. I call (1), (2) and (5), (6) *current-inflation* and *future-inflation* models, respectively. By comparing the two models, I observe that in the latter model, the output gap $\frac{dx_t}{dt}$ responds not only to inflation π_t but also to its changes $\frac{d\pi_t}{dt}$.

Two questions arise: i) How well does my continuous-time approximation (5), (6) represents the discrete-time model (3)–(4)? ii) How robust are the predictions of the stylized continuous-time model (1), (2) to the introduction of the new term $\frac{d\pi_t}{dt}$ in the IS curve (5)? These two questions are addressed in Sections 3 and 4, respectively.

3 The economy with a Taylor rule

In this section, I compare discrete- and continuous-time models in the absence of liquidity trap. First, I augment the model (5), (6) to include Taylor rule; second, I construct characteristic roots for such a model; third, I establish the parameter regions that lead to different types of characteristic roots; fourth, I derive closed-form solutions; and finally, I compare the implications of the continuous- and discrete-time models under several alternative monetary-policy scenarios.

3.1 Continuous-time model with the Taylor rule

Let me first consider a Taylor rule in discrete time which includes feedback to inflation, expected inflation and output gap:

$$i_t = i_t^* + \phi_\pi \pi_t + \phi_{E\pi} E_t \left[\pi_{t+1}\right] + \phi_y x_t + \varepsilon_t, \tag{7}$$

where i_t^* is the desired interest rate path; $\phi_{E\pi} \geq 0$; $\phi_\pi \geq 0$ and $\phi_y \geq 0$ are the coefficients in the Taylor rule, and ε_t is the interest rate shock. By using linear approximation $\pi_{t+1} \approx \pi_t + \frac{d\pi_t}{dt}$, I formulate a parallel rule for the continuous-time model:

$$i_t = i_t^* + \phi_\pi \pi_t + \phi_{E\pi} \left[\pi_t + \frac{d\pi_t}{dt}\right] + \phi_y x_t + \varepsilon_t. \tag{8}$$

By differentiating (6), substituting the result into (5), using the fact that $\left[\rho\pi_t - \frac{d\pi_t}{dt}\right] = k\left(x_t + g_t\right)$ and combining the result with the Taylor rule (8), I obtain the following second-order differential equation:

$$\frac{d^2\pi_t}{dt^2} - \left(\rho + k\sigma\left(1 - \phi_{E\pi}\right) + \sigma\phi_y\right)\frac{d\pi_t}{dt} - k\sigma\pi_t\left(1 - \phi_\pi - \phi_{E\pi} - \frac{\phi_y\rho}{k}\right)$$
$$= -k\sigma\left(i_t^* - i_t^n - \phi_y g_t\right) - k\frac{dg_t}{dt} + \sigma\phi_y g_t. \tag{9}$$

3.2 Characteristic roots

Let me re-write the second-order differential equation (9) in compact notation

$$\frac{d^2\pi_t}{dt^2} + a\frac{d\pi_t}{dt} + s\pi_t = -z_t, \tag{10}$$

where $a \equiv -\left(\rho + k\sigma\left(1-\phi_{E\pi}\right)+\sigma\phi_y\right)$; $s \equiv -k\sigma\left(1-\phi_\pi - \frac{\phi_y \rho}{k}\right)$; and z_t includes all exogenous variables, g_t, i_t^*, ε_t, i_t^n,

$$z_t \equiv k\sigma\left(i_t^* - i_t^n - \phi_y g_t\right) + k\frac{dg_t}{dt} - \sigma\phi_y g_t. \tag{11}$$

Below, I establish the properties of a homogeneous equation that corresponds to (10), $\frac{d^2\pi_t}{dt^2} + a\frac{d\pi_t}{dt} + s\pi_t = 0$.

Theorem 1 *The roots* $\lambda_{1,2} = \frac{-a\pm\sqrt{a^2-4s}}{2}$ *to characteristic equation* $\lambda^2 + a\lambda + s = 0$ *satisfy*

Case	*Restrictions on eigenvalues*	*Restrictions on parameters*
i)	$\lambda_1 > 0,\ \lambda_2 < 0$	$\phi_{E\pi} < \phi_{E\pi}^1$ *and* $\phi_{E\pi} > \phi_{E\pi}^4$
ii)	$\lambda_1 \geq 0,\ \lambda_2 \geq 0$	$\phi_{E\pi}^1 \leq \phi_{E\pi} < \phi_{E\pi}^2$ *and* $\phi_{E\pi}^3 \leq \phi_{E\pi} < \phi_{E\pi}^4$
iii)	$\lambda_1 = \lambda_2 = \lambda > 0$	$\phi_{E\pi} = \phi_{E\pi}^2$ *and* $\phi_{E\pi} = \phi_{E\pi}^3$
iv)	$\lambda_{1,2} = \mu \pm \eta\iota$ *with* $\mu > 0$	$\phi_{E\pi}^2 < \phi_{E\pi} < \phi_{E\pi}^3$,

where $\phi_{E\pi}^1 = -\phi_\pi - \frac{\rho\phi_y}{k}$, $\phi_{E\pi}^2 = \phi_{E\pi}^1 + \frac{1}{\sigma k}\left(\sqrt{\left(1+\sigma\phi_y\right)\left(1+\rho\right)+\phi_\pi\sigma k} - 1\right)^2$, $\phi_{E\pi}^3 = \phi_{E\pi}^1 + \frac{1}{\sigma k}\left(\sqrt{\left(1+\sigma\phi_y\right)\left(1+\rho\right)+\phi_\pi\sigma k} + 1\right)^2$, $\phi_{E\pi}^4 = \phi_{E\pi}^1 + \frac{2}{\sigma k}\left(\left(1+\sigma\phi_y\right)\left(1+\rho\right)+\phi_\pi\sigma k + 1\right)$.

Proof. Maliar and Taylor (2018) proved a similar theorem for the discrete-time model (3), (4), (7). While it is possible to apply their proof strategy to the continuous-time case, there is more simple proof, namely, I will show that in the limit, the continuous-time model (10) leads to the same characteristic equation and hence, the same characteristic roots as the discrete-time model (3), (4) and (7).

As a starting point, I construct a homogeneous equation for the discrete-time model (3), (4), (7):

$$\pi_{t+2} - \left[1 + \frac{1}{\beta} + \sigma\phi_y + \frac{\sigma\kappa\left(I - \phi_{E\pi}\right)}{\beta}\right]\pi_{t+1} + \left[\frac{\left(1+\sigma\phi_y\right)}{\beta} + \frac{\sigma\kappa\phi_\pi}{\beta}\right]\pi_t = 0. \tag{12}$$

I then re-write (12) as

$$\begin{aligned}
&\left(\pi_{t+2} - \pi_{t+1}\right) - \left(\pi_{t+1} - \pi_t\right) \\
&\quad - \pi_{t+1}\left(\frac{1}{\beta} - 1 + \frac{\sigma\kappa\left(I-\phi_{E\pi}\right)}{\beta} + \phi_y\sigma\right) + \pi_t\left(\frac{1}{\beta} - 1 + \frac{\sigma\kappa}{\beta}\phi_\pi + \frac{\sigma}{\beta}\phi_y\right) = \\
&\qquad \frac{d^2\pi_t}{dt^2} - \left(\rho + \sigma k\left(I - \phi_{E\pi}\right) + \phi_y\sigma\right)\frac{d\pi_t}{dt} - \sigma k\left(1 - \phi_\pi - \phi_{E\pi} - \frac{\phi_y\rho}{k}\right)\pi_t = 0,
\end{aligned}$$

where the last expression is the homogeneous equation for our continuous-time model (10). To get these results, I approximate the differences with derivatives $\frac{\pi_{t+1}-\pi_t}{(t+1)-t} \approx \frac{d\pi_t}{dt}$

and $\frac{\pi_{t+2}-\pi_{t+1}}{(t+2)-(t+1)} - \frac{\pi_{t+1}-\pi_t}{(t+1)-t} \approx \frac{\frac{d\pi_{t+1}}{dt}-\frac{d\pi_t}{dt}}{(t+1)-t} \approx \frac{d^2\pi_t}{dt^2}$, and I introduce notation $\rho \equiv \frac{1}{\beta} - 1$ and $k = \frac{\kappa}{\beta}$. Given the connection between discrete- and continuous-time models, I obtain the parameters regions for the continuous-time model (10) by simply changing notation in the statement of Theorem 1 in Maliar and Taylor (2018) and by using a notion of stability for the continuous-time case, namely, a root $|m| \geq 1$ is unstable in discrete time when the corresponding root is $\lambda \geq 0$ in continuous time, etc. ■

In case i), one root is stable and the other is unstable (explosive). In the remaining cases ii)-iv), both roots are unstable. I have further results for the special case when the monetary policy rule (7) contains only current but not future inflation.

Theorem 2 *Assume $\phi_{E\pi} = 0$. Then, the roots $\lambda_{1,2} = \frac{-a \pm \sqrt{a^2-4s}}{2}$ to characteristic equation $\lambda^2 + a\lambda + s = 0$ satisfy:*

Case	*Restrictions on eigenvalues*	*Restrictions on parameters*
i)	$\lambda_1 > 0$, $\lambda_2 < 0$	$\phi_\pi < \phi_\pi^1$
ii)	$\lambda_1 \geq 0$, $\lambda_2 \geq 0$	$\phi_\pi^1 \leq \phi_\pi < \phi_\pi^2$
iii)	$\lambda_1 = \lambda_2 = \lambda > 0$	$\phi_\pi = \phi_\pi^2$
iv)	$\lambda_{1,2} = \mu \pm \eta\iota$ *with* $\mu > 0$	$\phi_\pi > \phi_\pi^2$

where $\phi_\pi^1 = 1 - \frac{\rho\phi_y}{k}$ and $\phi_\pi^2 = \phi_\pi^1 + \frac{1}{4\sigma k}\left(\rho + \sigma\phi_y + \sigma k\right)^2$.

Proof. This result for our continuous-time model (10) also follows from the corresponding result for the discrete-time model (3), (4), (7) established in Maliar and Taylor (2018). I simply change notation to $\rho \equiv \frac{1}{\beta} - 1$ and $k = \frac{\kappa}{\beta}$. ■

The model in continuous time has one stable and one unstable root when the response of the monetary authority to inflation $\phi_\pi < \phi_\pi^1$ is weak; both roots become unstable $\phi_\pi^1 \leq \phi_\pi < \phi_\pi^2$ when the response to inflation becomes stronger; the roots become repeated and unstable and finally when the response to inflation reaches the threshold level ϕ_π^2; finally, the roots are complex and unstable, when $\phi_\pi > \phi_\pi^2$.

3.3 Closed-form solutions

Below, I show closed-form solutions of the model under different types of characteristic roots (real distinct, real repeated and complex).

Theorem 3 *The solution to the new Keynesian model (10) for cases i)-iv) in Theorem 1 is given by:*
i). For two distinct real roots such that root λ_1 is unstable and root λ_2 is stable, i.e., $\lambda_1 > 0$ and $\lambda_2 < 0$, I have

$$\pi_t = C_1 e^{\lambda_1 \cdot t} + C_2 e^{\lambda_2 \cdot t} + \frac{1}{\lambda_1 - \lambda_2}\left[\int_{s=t}^{\infty} e^{\lambda_1 \cdot (t-s)} z_s ds + \int_{s=-\infty}^{t} e^{\lambda_2 \cdot (t-s)} z_s ds\right]; \tag{13}$$

ii) For two distinct real roots $\lambda_1 \neq \lambda_2$ that are both unstable $\lambda_1 \geq 0$, $\lambda_2 \geq 0$, I have

$$\pi_t = C_1 e^{\lambda_1 \cdot t} + C_2 e^{\lambda_2 \cdot t} + \frac{1}{\lambda_1 - \lambda_2}\left[\int_{s=t}^{\infty} e^{\lambda_1 \cdot (t-s)} z_s ds - \int_{s=t}^{\infty} e^{\lambda_2 \cdot (t-s)} z_s ds\right]; \tag{14}$$

iii). For two repeated real roots $\lambda_1 = \lambda_2 = \lambda$ that are unstable $\lambda \geq 0$, I have:

$$\pi_t = C_1 e^{\lambda_1 \cdot t} + C_2 t e^{\lambda_2 \cdot t} + \left[\int_{s=t}^{\infty} e^{\lambda \cdot (t-s)} z_s ds - \int_{s=t}^{\infty} s e^{\lambda \cdot (t-s)} z_s ds\right]; \tag{15}$$

iv). For complex roots $\lambda_{1,2} = \mu \pm \eta\iota$ that are unstable, i.e., $\mu \geq 0$, I have:

$$\pi_t = C_1 e^{\mu t} \sin(\eta t) + C_2 e^{\mu t} \cos(\eta t) + \frac{1}{\eta}\int_{s=t}^{\infty} e^{\mu \cdot (t-s)} \sin((t-s)\eta) z_s ds; \tag{16}$$

where C_1, C_2 in (13)–(16) are arbitrary constants.

Proof. The solution to (10) is given by the sum of a general solution to a homogeneous equation $\frac{d^2\pi_t}{dt^2} + a\frac{d\pi_t}{dt} + s\pi_t = 0$ and a particular solution satisfying the non-homogeneous equation (10). Homogeneous second-order difference equations with constant coefficients are well studied in the field of differential equations; the solutions to such equations are the parts of expressions (13)–(16) that contain integration constants C_1 and C_2. Therefore, the contribution of the present paper consists in constructing solutions to non-homogeneous equations, i.e., the remaining parts of (13)–(16). The fact that the particular solutions to non-homogeneous equations, i.e., the remaining parts of (13)–(16), satisfy the non-homogeneous equation can be verified directly, by substituting them into (10). ■

To make the entire solutions (14)–(16) forward stable, I must set integration constants at $C_1 = 0$ and $C_2 = 0$. An exception is the case i) in which forward stability is consistent with any integration constant C_2 on the stable root in equation (13). Therefore, a forward-stable solution is unique in cases ii)-iv). and it is indeterminate in case i). The last case is studied in Cochrane (2017) in the context of his continuous-time model. He derives closed-form solutions to the model with one stable and one unstable root (13), which corresponds to case i) of Theorems 1, 2 and 3. Thus, the novel part of Theorem 3 is closed-form solutions (14)–(16) under the other types of characteristic roots.

Finally, the results in Theorems 1-3 are established for the future-inflation model (5), (6). With the following corollary, I extend these results to the case of the stylized continuous-time model (1), (2) of Werning (2012) and Cochrane (2017).

Corollary 4 *The economy (1), (2), (8) under $\phi_{E\pi} = 0$ in the Taylor rule (7) is described by the same difference equation (10) except that the coefficient $a = -\left(\rho + k\sigma\left(1 - \phi_{E\pi}\right) + \sigma\phi_y\right)$ changes to $a \equiv -\left(\rho + \sigma\phi_y\right)$.*

Therefore, Theorems 2 and 3 apply directly to the stylized continuous-time model (1), (2), except that the characteristic roots will be different. Theorem 1 places restrictions on $\phi_{E\pi}$, and it is not applicable to this version of the model.

3.4 Solution with one anticipated shock

To compare the predictions of the discrete- and continuous-time models, I analyze their impulse-response functions.

Discrete-time model. Let me first construct impulse-response functions of the discrete-time economy (3), (4), (7) to a single anticipated shock. I assume that the disturbance term in (7) is $\varepsilon_t = 0$ for all t except of period T in which the economy faces a shock $\varepsilon_T > 0$. The results of Theorem 3 in Maliar and Taylor (2018) make it possible to obtain the impulse-response functions in a closed form.

Case i): for two distinct real roots such that one root is unstable $|m_1| \geq 1$ and the other root is stable $|m_2| < 1$, I have

$$\begin{aligned} t &\leq T,\ \pi_t = C_2 m_2^t + \frac{\kappa\sigma\varepsilon_T}{\beta(m_1 - m_2)}\left[m_1^{t-1-T}\right], \\ t &> T,\ \pi_t = C_2 m_2^t + \frac{\kappa\sigma\varepsilon_T}{\beta(m_1 - m_2)}\left[m_2^{t-1-T}\right]. \end{aligned} \tag{17}$$

Case ii): for two distinct unstable roots $|m_1| \geq 1$, $|m_2| \geq 1$, for $t \leq T$, I have

$$\pi_t = \frac{\kappa\sigma\varepsilon_T}{\beta(m_1 - m_2)}\left[m_1^{t-1-T} - m_2^{t-1-T}\right]. \tag{18}$$

Case iii): for two identical unstable roots $m_1 = m_2 = m$, such that $|m| > 0$, for $t \leq T$, I have

$$\pi_t = \frac{\kappa\sigma\varepsilon_T}{\beta m}\left[(t-1)\, m^{t-1-T} - T m^{t-1-T}\right]. \tag{19}$$

Case iv): for unstable complex roots $\lambda_{1,2} = \mu \pm \eta\iota$ such that $r \equiv \sqrt{\mu^2 + \eta^2} \geq 1$, for $t \leq T$, I have

$$\pi_t = \frac{\kappa\sigma\varepsilon_T}{\beta\eta} r^{t-1-T} \sin\left(\theta\left(t - 1 - T\right)\right). \tag{20}$$

where $\theta \equiv \arctan\left(\frac{\eta}{\mu}\right)$. For the cases ii)-iv), for $t > T$, I have $\pi_t = 0$.

Continuous-time model. Let me show parallel impulse response functions to an anticipated shock in the continuous-time model (5), (6) and (8). To model the impulse shock in continuous time, I use a Dirak delta function – a normalized function on the real line which is zero everywhere except at zero, where it is infinite, i.e., $\delta(x) = \begin{cases} +\infty, & x = 0 \\ 0, & x \neq 0 \end{cases}$

and $\int_{-\infty}^{+\infty} \delta(x)\, dx = 1$. A useful property of the delta function is its shifting in time, namely, for any Lebesgue integrable function f, I have $\int_{-\infty}^{+\infty} f(t)\, \delta(t-T)\, dt = f(T)$. Let me assume that the economy gets an impulse shock ε at T given by a delta function, i.e., $i_t - i_t^n = \varepsilon\delta(t-T)$, which implies $z_t = \kappa\sigma(i_t - i_t^n) + \kappa\frac{dg_t}{dt} = \kappa\sigma\varepsilon\delta(t-T)$ (like in the discrete-time economy, I keep g_t constant for all t so that $\frac{dg_t}{dt} = 0$). Below, I provide the impulse-response functions for the four cases established in Theorem 3.

Case i): two distinct real roots such that one root is unstable $\lambda_1 \geq 0$ and the other root is stable $\lambda_2 < 0$:

$$\begin{aligned} t &\leq T,\ \pi_t = Ce^{\lambda_2 \cdot t} + \frac{k\sigma\varepsilon_T}{\lambda_1 - \lambda_2} e^{\lambda_1 \cdot (t-T)}, \\ t &> T,\ \pi_t = Ce^{\lambda_2 \cdot t} + \frac{k\sigma\varepsilon_T}{\lambda_1 - \lambda_2} e^{\lambda_2 \cdot (t-T)}. \end{aligned} \tag{21}$$

Case ii): two distinct unstable roots $\lambda_1 \geq 0$, $\lambda_2 \geq 0$, for $t \leq T$:

$$\pi_t = \frac{k\sigma\varepsilon_T}{\lambda_1 - \lambda_2}\left[e^{\lambda_1(t-T)} - e^{\lambda_2(t-T)}\right]. \tag{22}$$

Case iii): two identical unstable roots $\lambda_1 = \lambda_2 = \lambda > 0$, for $t \leq T$:

$$\pi_t = \kappa\sigma\varepsilon_T\left[te^{\lambda(t-T)} - Te^{\lambda(t-T)}\right]. \tag{23}$$

Case iv): unstable complex roots $\lambda_{1,2} = \mu \pm \eta\iota$ with $\mu \geq 0$, for $t \leq T$:

$$\pi_t = \frac{\kappa\sigma\varepsilon_T \upsilon}{\eta} e^{\mu \cdot (t-T)} \sin\left((t-T)\,\eta\right). \tag{24}$$

For cases ii)-iv), for $t > T$, I have $\pi_t = 0$.

Comparison between the discrete- and continuous-time responses. The impulse-response functions (21)–(24) for the continuous-time model have the same structure as those (17)–(20) in the discrete-time model (recall that $k \equiv \frac{\kappa}{\beta}$, so I would not have β in (17)–(20) if I express both of them in terms of k). There are two differences between the two solutions. The first one is a time shift, namely, in the discrete-time model, π_t reaches zero in the next period after the shock, i.e., at $t = T + 1$, while in the continuos-time model, it reaches zero right in the moment of the shock. In other words, π_t is shifted by one period ahead in the former model relatively to the latter model. The second difference is that in the discrete-time model, I have a decay/explosion determined by the root like m^{t-1-T}, while in the continuous-time model, I have exponential functions like $e^{\lambda \cdot (t-T)}$.

To see how close the two models are to each other, in Figure 1, I analyze the model the model's implications under three alternative parameterizations: 1) $\phi_{E\pi} = 1$, $\phi_\pi = 0$ and $\phi_y = 0$; 2) $\phi_{E\pi} = 0$, $\phi_\pi = 1$ and $\phi_y = 0$; 3) $\phi_{E\pi} = 0$, $\phi_\pi = 2$ and $\phi_y = 0.5$ (I use $\kappa = 0.11$, $\sigma = 1$ and $\beta = 0.99$). A negative interest-rate shock is announced to happen in period 30.

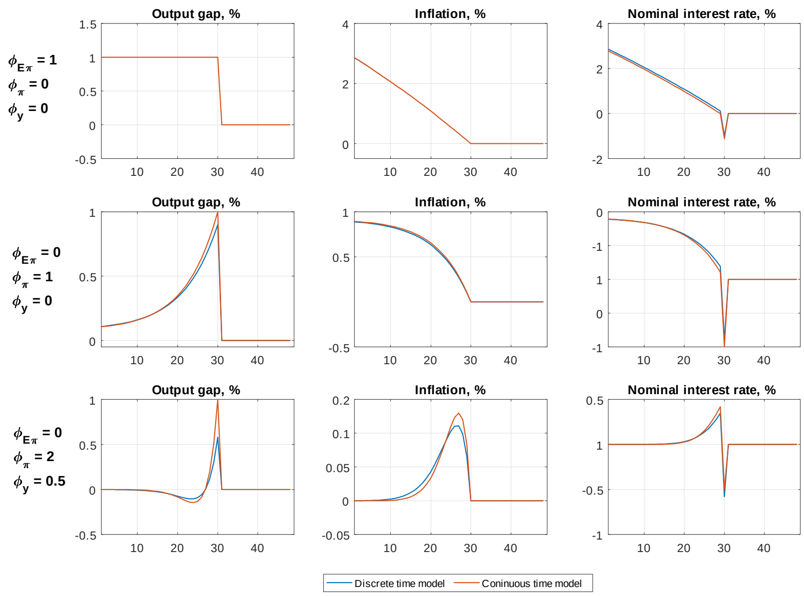

Figure 1. Comparison of responses of continuous- and discrete-time models under three alternative parameterizations.

The first parameterization leads to the so-called FG puzzle that consists in an immediate reaction of both output and inflation in response to the announced future interest-rate shock; see Del Negro et al. (2015) and McKay et al. (2016. The second parameterization leads to a weaker form of the FG puzzle when only inflation reacts immediately. The last parameterization corresponds to an empirically relevant version of the Taylor rule. (In this figure, I show lagged inflation for the discrete-time model). There are quantitative differences between the two models, in particular, under the last parameterization because my approximation of differences with derivatives is not exact. However, these differences are relatively small and impulse responses functions of both models are qualitatively similar. I performed a number of further experiments, and I observed that the continuous-time model (5), (6) with future inflation provides a fairly good approximation of the discrete-time model (3), (4). On that basis, I conclude that the continuous-time model with future inflation is a valid approximations of the discrete-time model.

4 Liquidity trap revisited

Cochrane (2017) considers an economy (1), (2) that faces a zero lower bound (ZLB) on nominal interest rates. In the presence of active ZLB, the monetary authority cannot use Taylor-style rules and chooses instead a discretionary path for the interest rate by setting i_t equal to the desired level i_t^*. To characterize the solution, Cochrane (2017) represents the model by a second-order difference equation. For the sake purpose of my analysis, I will formulate the second-order difference equation that describes both the current- and future-inflation models, (1), (2) and (5), (6), respectively. In what follows, I will interpret these two models as continuous-time and discrete-time versions of new-Keynesian model, respectively.

4.1 New Keynesian model with liquidity trap

By differentiating (6), substituting the result into (5), using the fact that $\left[\rho\pi_t - \frac{d\pi_t}{dt}\right] = k\left(x_t + g_t\right)$, I get

$$\frac{d^2\pi_t}{dt^2} - \left(\rho + Ik\sigma\right)\frac{d\pi_t}{dt} - k\sigma\pi_t = -z_t, \tag{25}$$

where $z_t \equiv k\sigma\left(i_t^* - i_t^n\right) + k\frac{dg_t}{dt}$, and $I \in \{0,1\}$ is an indicator function. By setting $I = 0$ and $I = 1$, I obtain the current- and future-inflation models, respectively. Thus, all the difference between the two models is captured by an additional coefficient $-k\sigma$ on $\frac{d\pi_t}{dt}$.

A forward-stable solution to (25) the current-inflation model and future-inflation models are constructed in Cochrane (2017) and case i) of Theorem 3, respectively. The solutions to the two models are described by the same formula,

$$\pi_t = Ce^{\lambda^b \cdot t} + \frac{1}{\lambda^f + \lambda^b}\left[\int_{s=-\infty}^{t} e^{-\lambda^b\cdot(t-s)} z_s ds + \int_{s=t}^{\infty} e^{-\lambda^f\cdot(s-t)} z_s ds\right], \tag{26}$$

however the characteristic roots in the future-inflation model are different from those in the current-inflation model by the $Ik\sigma$ term:

$$\lambda^f = \frac{1}{2}\left(\rho + Ik\sigma + \sqrt{\left(\rho + Ik\sigma\right)^2 + 4k\sigma}\right), \quad \lambda^b = \frac{1}{2}\left(\rho + Ik\sigma - \sqrt{\left(\rho + Ik\sigma\right)^2 + 4k\sigma}\right). \tag{27}$$

In turn, the Phillips curve is the same in both models, so that the output gap is computed from $kx_t = -kg_t + \rho\pi_t - \frac{d\pi_t}{dt}$.

According to (26), π_t is determined by two components: a forward-looking (FL) component $\int_{s=t}^{\infty} e^{-\lambda^f\cdot(s-t)} z_s ds$ that embodies the effect of the future shocks on inflation and a backward-looking component $\int_{s=-\infty}^{t} e^{-\lambda^b\cdot(t-s)} z_s ds$ that embodies the effect of the past shocks; the characteristic roots λ^f and λ^b determine the rates at which the impact of the future and past shocks are discounted, respectively. In other words, the roots determine how quickly the economy is stabilized after the shocks.

How different can be characteristic roots in the current- and future-inflation models? To make a simple calculation, I assume that ρ is 0.01 that corresponds to quarterly data, k varies from 1 to 20 as in Cochrane (2017); and σ varies from 0.1 to 10. In the current-inflation model, ρ is small relatively to $4k\sigma$, so the roots are approximately symmetric and equal to $\lambda^f \approx \sqrt{k\sigma}$ and $\lambda^b \approx -\sqrt{k\sigma}$; for example, if $k\sigma = 0.1$, I have $\lambda^f \approx 0.3$ and $\lambda^b \approx -0.3$, while if $k\sigma = 200$, I have $\lambda^f \approx 14$ and $\lambda^b \approx -14$. However, in the future-inflation model, the roots critically depend on the value of $k\sigma$. Namely, when $k\sigma$ is small, the term $4k\sigma$ dominates $(\rho + Ik\sigma)^2$, so that the roots are approximately the same in the current- and future-inflation models; however, when $k\sigma$ is large, the roots become highly asymmetric and approximately equal to $\lambda^f \approx k\sigma$ and $\lambda^b \approx -1$. That is, the unstable root is considerably more explosive forward and the stable root is less explosive backward in the future-inflation model than in the current-inflation model.

What is the intuition the asymmetricity of roots in the future-inflation model? Such model has two channels of transmission of inflationary shocks, one is the inflation itself π_t ad the other is its inflationary expectations $\frac{d\pi_t}{dt}$. In contrast, the current-inflation model has just one channel π_t. When the two channels work in the same direction, the impact of inflation on output is reenforced and otherwise, it is weakened. For example, suppose that the economy is initially in the steady state and that it faces a negative interest-rate shock in the future, i.e., $i_t < i_t^n$ for some t. The IS curve (5) implies $\frac{dx_t}{dt} < 0$ and $x_t < 0$. In turn, the Phillips curve (6) implies that there is deflation at present $\pi_t < 0$ and the expectation of deflation in the future $\frac{d\pi_t}{dt} < 0$, both of which decrease x_t even further. This is why the unstable root λ^f is larger, and the associated forward explosion is stronger in the future-inflation model than in the current-inflation model. On the contrary, in the case of the backward explosion, inflation and inflationary expectations act in opposite directions, which reduces the size of the stable root λ^b, and dampens the associated backward explosion.

Summarizing, let me highlight two key implications of the above analysis:

Implication 1. Cocharane (2017) shows that the liquidity trap produces a multiplicity of forward-stable equilibria and the backward explosion in the new Keynesian model. This insight is not affected by whether the model contains the current or future inflation: by (27), both the current- and future-inflation models have one unstable and one stable roots, $\lambda^f > 0$ and $\lambda^b < 0$, respectively.

Implication 2. The future-inflation model has an additional mechanism of transmitting inflationary shocks to output – inflationary expectation. As a result, roots λ^f and λ^b differ dramatically in the two models under large $k\sigma$. Potentially, this additional mechanism can lead to significantly different quantitative implications. We assess the difference between the two models in the next section.

4.2 Numerical comparison of the current- and future inflation models

In the initial interval of time (namely, at $t \in [0, 5]$), the economy experiences a liquidity trap due to a negative shock to the natural rate of interest. During this period, the nominal interest rate is zero. As the negative shock is over, the central bank commits to keeping rates at zero (the FG policy) during the subsequent τ periods (i.e., at $t \in [5, 5 + \tau]$). Following Cochrane (2017), I consider two types of equilibria: a no-jump equilibrium, in which inflation is initially in steady state, i.e., $\pi_0 = 0$ (see his Figure 7) and a standard equilibrium, in which inflation arrives to steady state at the end of FG policy, i.e., $\pi_{T+\tau} = 0$ (see his Figure 6). In the main text, I assume moderately large risk-aversion coefficient $\sigma = 5$ in which the differences between the current- and future-inflation models are more pronounced; the case of low-risk aversion coefficient $\sigma = 1$ is analyzed in Appendix A. For the sake of a meticulous comparison with Cochrane (2017), I constructed my numerical solutions using his code as a basis. In Sections 4.2.1, 4.2.2 and 4.2.3, I illustrate the effects of the new term $k\sigma$ on FG, government multipliers, and liquidity trap, respectively.

4.2.1 Forward guidance

I show the results for four alternative scenarios: i) no FG; ii) standard equilibrium; iii) no-jump equilibrium; iv) extended FG scenarios. Under the first scenario, I set $\tau = 0$; under the second and third scenarios, I found τ to satisfy a zero terminal condition $\pi_{T+\tau} = 0$ and a no-jump condition $\pi_0 = 0$, respectively; under the last scenario, I extended τ by 15 percent relative to the standard equilibrium (I chose this percentage point to match the experiments in Cochrane, 2017, shown in his Figures 6 and 7).

Current-inflation model. In Figure 2, I show the results for the current-inflation model under a relatively high flexibility of prices, $k = 5$; and in Appendix A, I show the results for less flexible prices, $k = 1$.

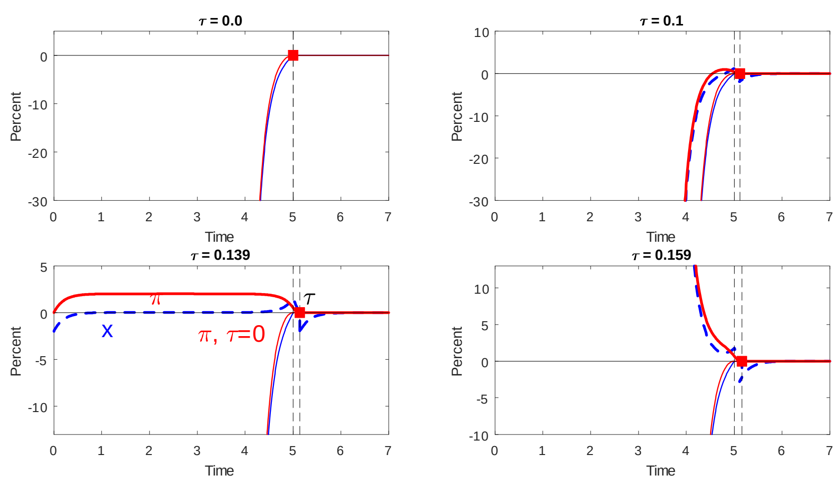

Figure 2. FG scenarios in the current-inflation model under $k = 5$, $\sigma = 5$: i) no FG, ii) standard equilibrium ($\pi_{T+\tau} = 0$) iii) no-jump equilibrium ($\pi_0 = 0$) and extended FG.

The main features of the equilibrium in the current-inflation model are as follows: First, there is a strong downward backward explosion of the output gap and inflation in the standard equilibrium in the absence of FG; see panel $\tau = 0$. Second, the implications of the model are extremely sensitive to the duration of FG. Namely, FG with $\tau = 0.1$ visibly reduces the backward explosion; a slight increase in the FG duration to $\tau = 0.139$ dampens the downward backward explosion and produces no-jump equilibrium; and a further increase in the FG duration to $\tau = 0.159$ brings an opposite extreme – a strong upward backward explosion. Finally, after the end of FG, we observe a reversal in the output-gap dynamics that was first reported in Carlstrom et al. (2015). Given that the period it is calibrated to is one quarter, we conclude that three additional days of FG are sufficient to entirely change the properties of the equilibrium. Such an excessive sensitivity of the model to FG appears to be empirically implausible.

Future-inflation model. I now perform the same experiments with the future-inflation model; see Figure 3. Again, I show the case of $k = 5$ in the main text and present the case of $k = 1$ in Appendix A.

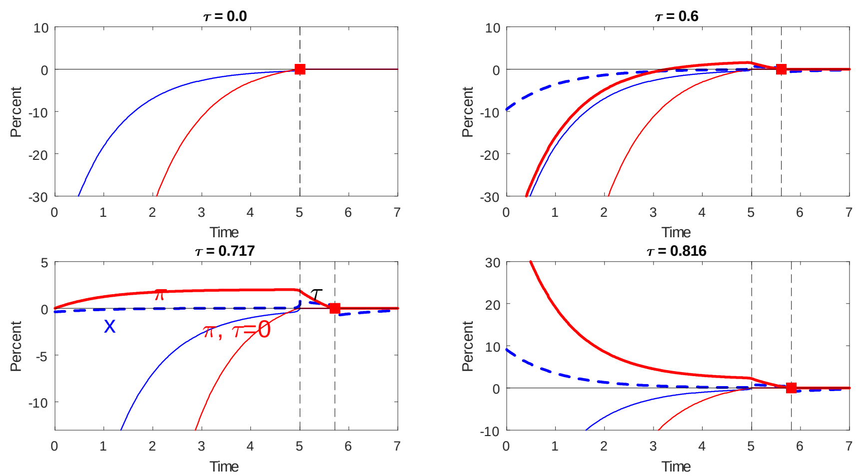

Figure 3. FG scenarios in the future-inflation model under $k = 5$, $\sigma = 5$: i) no FG, ii) standard equilibrium ($\pi_{T+\tau} = 0$) iii) no-jump equilibrium ($\pi_0 = 0$) and extended FG.

Not surprisingly, the qualitative implications of the future-inflation model are similar to those of the current-inflation model. Again, we observe backward explosion in the absence of FG; the size of backward explosion gradually reduces and the economy switches to no-jump equilibrium when the duration of FG increases; and eventually, the economy reverses from a backward to forward explosion. However, the quantitative expression of the above effects is dramatically different. First, the FG duration must be almost 10 times larger in the future-inflation model than in the current-inflation model in order to produce comparable effects on the solution. Second, the size of the backward explosion is orders of magnitudes smaller in the future-inflation model than in the current inflation model.

The magnitude of the FG effect on the equilibrium is a critical issue in the literature. Del Negro et al. (2015) and McKay et al. (2016) coined the FG puzzle, which is implausibly large sensitivity of the new Keynesian model to future shocks; see also Maliar and Taylor (2018) for related analysis and the literature review. However, in the presence of liquidity trap, there is a multiplicity of equilibrium and the size of the FG effects depends on which equilibrium is selected. One of possible equilibrium selection criteria is to focus on an equilibrium with a less dramatic sensitivity ("pick equilibria in which small frictions have small effects"). This criteria led Cochrane (2017) to select the no-jump equilibrium as more empirically plausible as it implies relatively mild inflation and small variations in the output gap, unlike the standard equilibrium. However, the effects of FG on the solution are dramatically reduced in the future-inflation model, compared to the current-inflation model. It could happen that the standard equilibrium passes a selection test in the former model, while it fails to pass such a test in the latter model.

4.2.2 Multipliers

I interpret the variable g_t as government spending; other possible interpretations are a technological regress and a deliberate destruction of capital; see Cochrane (2017) for a discussion. I assume that during the liquidity-trap period (i.e., at $t \in [0,5]$), government spending is strictly positive and constant, $g_t = g$, and zero afterwards, $g_t = 0$. The government multiplier is defined as $\frac{\partial x_t}{\partial g_t}$, evaluated at $g_t = 0$.

The effect of government spending on output is transmitted via future inflation. A positive government spending shock decreases both $\frac{d\pi_t}{dt}$ and π_t via the Phillips curve (6) which is the same in the current- and future-inflation models. However, the effect of government spending on output differs in the two models. In the current-inflation model, the reduction in π_t unambiguously increases x_t via the IS curve (1). However, in the future-inflation model, inflation π_t and inflationary expectations $\frac{d\pi_t}{dt}$ act in the opposite directions in the backward explosive solution, so their total effect on the output depends on the relative strength of the two channels. In fact, our numerical experiments show that under large σ's, the derivative effect overweighs the inflation effect, so that the IS curve (5) implies a negative effect of government spending on today's output.

Current-inflation model. Figure 4 shows the multipliers in a no-jump equilibrium ($\pi_0 = 0$) and standard equilibrium ($\pi_T = 0$) for the current-inflation model under $\sigma = 5$.

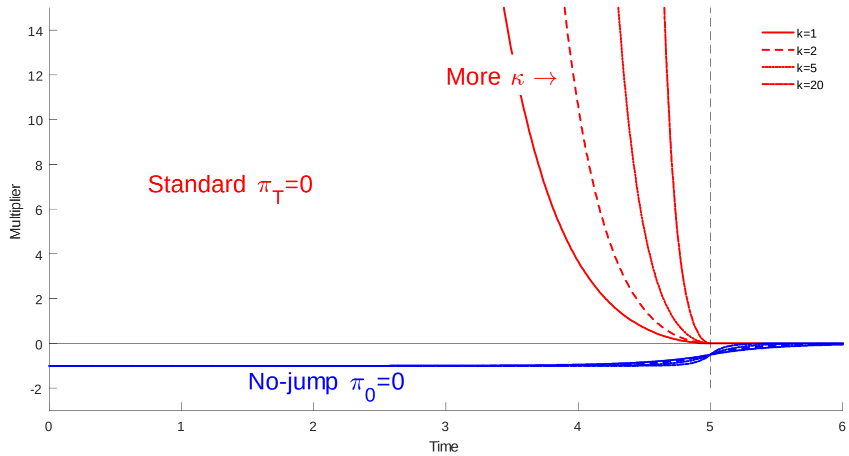

Figure 4. Multipliers in the current-inflation model under $\sigma = 5$ and $k = 1, 2, 5, 20$.

It turned out that under the no-jump equilibrium ($\pi_0 = 0$), the multipliers of government spending are close to -1, and there is practically no differences in the predictions when k varies (see blue lines). The multipliers are significantly larger than unity. In fact, they are explosive backward: they are huge in the beginning of the time horizon, and they decrease exponentially as the liquidity-trap period ends and government spending becomes equal to zero. Moreover, as the degree of price stickiness goes down (k increases), the multipliers increase.

The regularities observed in Figures 4 under $\sigma = 5$ are also observed under $\sigma = 1$; I do not show the last case because it is shown in Cochrane (2017); see his Figure 5. As is argued in that paper, larger multipliers and severer depressions for larger k constitute a policy paradox. That is, if government implements a policy to reduce price stickiness (to increase k), it will achieve a counterintuitive outcome: the economy's output will suffer more but the effectiveness of fiscal stimulus will increase. My analysis shows that an increase in σ will increase the multipliers even further and will make the depressions even worse, making this paradox even more dramatic.

Future-inflation model. In Figure 5, I show the multipliers in the future-inflation model under $\sigma = 5$.

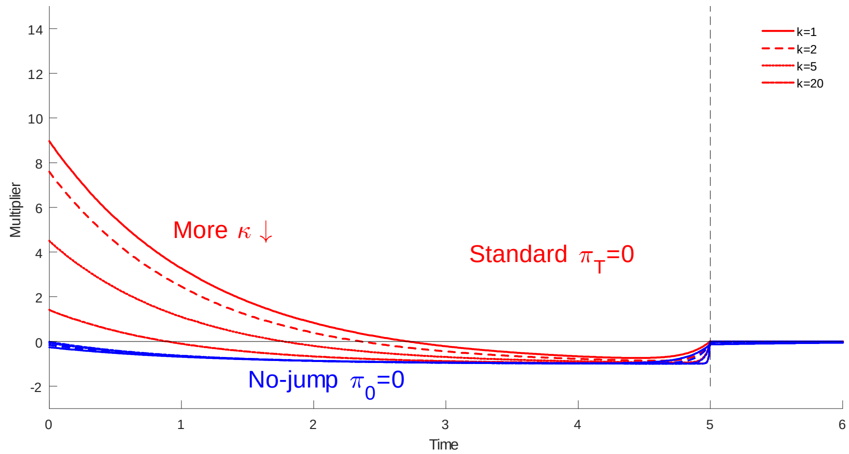

Figure 5. Multipliers in the future-inflation model: standard equilibrium ($\pi_T = 0$) in the alternative parameterization ($\sigma = 5$).

Like the FG effect, the backward explosion of multipliers is dramatically reduced in the future-inflation model compared to the current-inflation model. In the former model, the multipliers are substantially lower than in the latter model. In fact, in the beginning of the liquidity-trap period, they range from 1.5 for $k = 20$ to 9 for $k = 1$, which appears to be a reasonable range, at least for large values of k. Moreover, as the degree of price stickiness goes down (k increases), the multipliers decrease in the future-inflation model contrary to what we had in the current inflation model. This is because large k increases the size of the derivative effect, according to the IS curve (5). We observe that the closer is the economy to perfectly flexible prices, the less effective a fiscal stimulus is, i.e., the proposed modification helps resolve the undesirable implication of the original continuous-time model.

4.2.3 Liquidity trap

Liquidity trap is referred to a situation when the nominal interest rate is equal to zero (in our case, it is zero because of a negative shock to the natural rate of interest).

Current-inflation model. In Figure 6, I show the liquidity-trap scenario in the current-inflation model studied in Cochrane (2017):

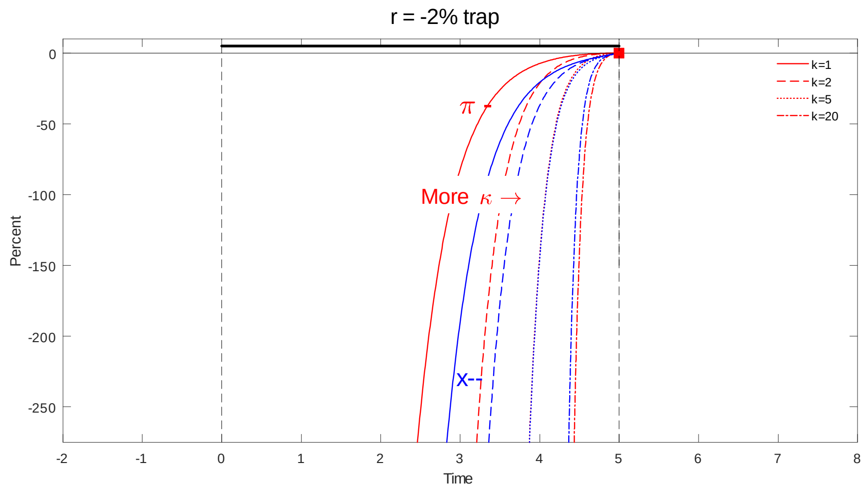

Figure 6. Liquidity trap in the current-inflation model: standard equilibrium ($\pi_T = 0$) under alternative parameterization ($\sigma = 5$).

In the figure, a lower degree of price stickiness (i.e., a higher value of k) leads to a severer depression (and deflation), which is counterintuitive. In the liquidity-trap scenario, the larger is the value of k, the larger is the output fall. That is, the closer is the economy to perfectly flexible prices, the larger is a negative effect of the liquidity trap. This is at odds with economic intuition: perfectly flexible prices should imply absent effects of monetary policy.

Future-inflation model. I next show the liquidity-trap situation in the future-inflation model in Figure 7.

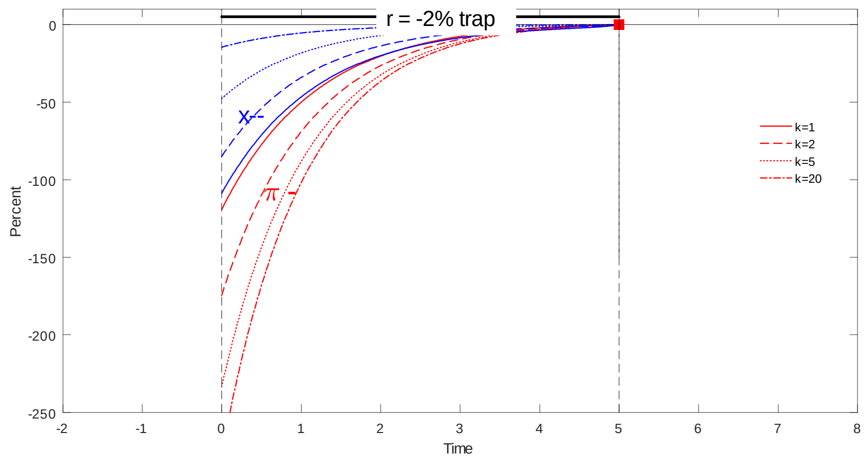

Figure 7. Liquidity trap in the future-inflation model: standard equilibrium ($\pi_T = 0$) under alternative parameterization ($\sigma = 5$).

Importantly, as prices become less rigid (k increases), the depressions are less severe, in contrast to the current-inflation model. As we argued earlier, this is because the inflation and derivative effects act in the opposite direction, and the derivative effect prevails. Therefore, there is no policy paradox in our future-inflation model: the government's efforts to reduce price stickiness leads to a logical outcome of reducing the magnitude of the depression caused by the liquidity trap.

5 Concluding remarks

The FG puzzle, policy paradoxes and other counterintuitive implications of the baseline new Keynesian model had motivated the literature to search for new directions in which the baseline model can be improved. Some of the directions highlighted in the literature are realistic demography (Del Negro et al., 2015), idiosyncratic household risk and borrowing constraints (McKay et al., 2016), monetary policy uncertainty (Husted et al., 2017), heterogeneous agents (Kaplan et al., 2017), bounded rationality (Gabaix, 2017). The goal of the present paper is not to claim that the big issues targeted by that literature can be resolved by merely switching between discrete and continuous time. Rather, the goal is to trace a sharper boundary between what the baseline model can and cannot explain. A deeper understanding of equilibria in the baseline model can help channel research efforts into those directions in which improvements are needed most. In that sense, the analytical results, closed-form solutions and sensitivity analysis provided in the present paper can be a useful reference point for the literature that aims at constructing more realistic and empirically relevant new Keynesian models.

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Online appendix A. Additional numerical experiments

This section includes additional results on the FG experiment. I consider benchmark parameterization $k = 1$, $\sigma = 1$, as well as some additional experiments for alternative parameterization $k = 5$, $\sigma = 5$.

The no-jump equilibrium ($\pi_0 = 0$) under $k = 1$, $\sigma = 1$

In this section, I display the no-jump equilibrium under benchmark parameterization ($k = 1$, $\sigma = 1$).

Current-inflation model. Under the no-jump equilibrium in Cochrane's (2017) current-inflation model, maintaining the interest rate at zero for additional τ periods makes almost no difference on inflation and the output gap dynamics; see Figure A1 (the same as Figure 7 in Cochrane (2017), except that I add here a plot of the output gap). As we can see, the FG policy has a positive effect on output but this effect is very small.

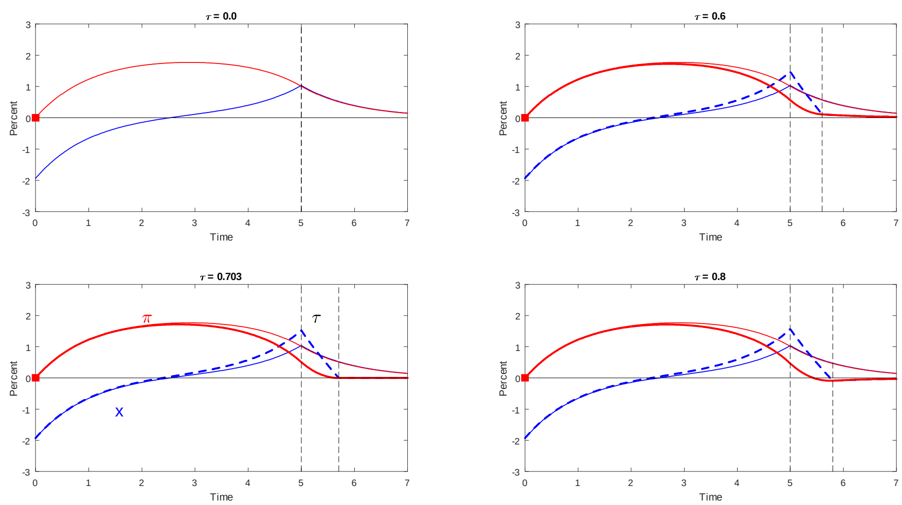

Figure A1. The current-inflation model: no-jump equilibrium ($\pi_0 = 0$) under benchmark parameterization ($k = 1$, $\sigma = 1$).

Future-inflation model. The no-jump equilibrium in the benchmark parameterization ($k = 1$, $\sigma = 1$) of the modified, future-inflation model is shown in Figure A2.

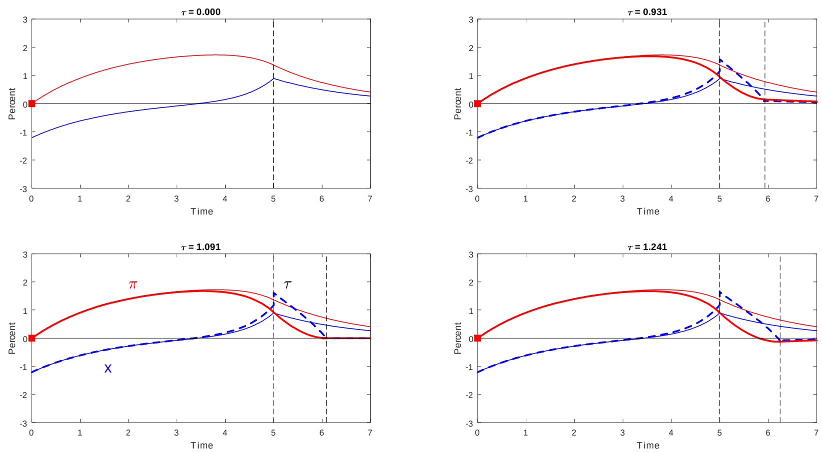

Figure A2. The future-inflation model: no-jump equilibrium ($\pi_0 = 0$) under benchmark parameterization ($k = 1$, $\sigma = 1$).

The future-inflation model has very similar dynamics; see Figure A2. The only sizable difference is in the length of the FG period τ to match both conditions $\pi_0 = 0$ and $\pi_{T+\tau} = 0$. In particular, the current-inflation model implies that in order to bound initial and post-FG jumps in inflation, the central bank should maintain the FG policy for $\tau = 0.703$, while the future-inflation model predicts a somewhat larger value $\tau = 1.091$.

The no-jump equilibrium was selected by Cochrane (2017) as being the most plausible one, as it implies mild inflation and small variations in the output gap. Remind that it was shown in the main text that the mildness of the implications disappears under alternative model's parameterizations, i.e., $k = 5$, $\sigma = 5$.

The no-jump equilibrium ($\pi_0 = 0$) under $k = 5$, $\sigma = 5$

In this section, I focus on the case of alternative parameterization, $k = 5$ and $\sigma = 5$. Relative to the main text, I provide here the results for additional values of τ. In Figures A3 and A4, the left bottom panels correspond to the left bottom panels in Figures 2 and 3, respectively.

Current-inflation model. The results for the current-inflation model are reported in Figure A3.

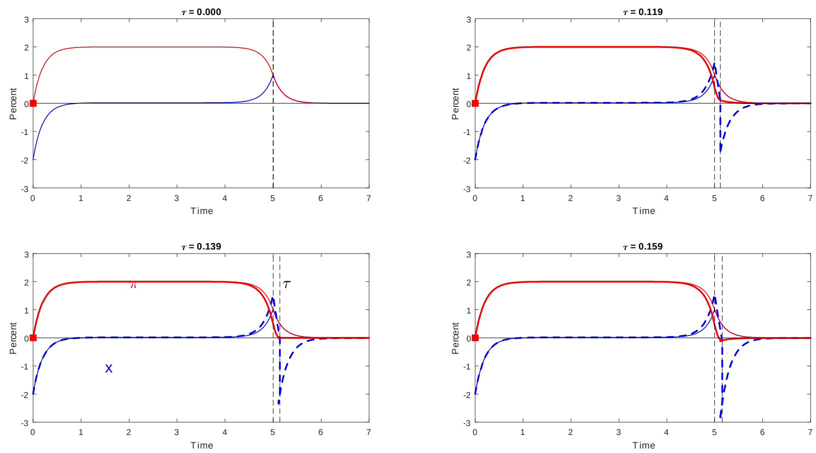

Figure A3. The current-inflation model: a no-jump equilibrium ($\pi_0 = 0$) under alternative parameterization ($k = 5$, $\sigma = 5$).

To be specific, the current-inflation model predicts that FG is detrimental for output: after the FG policy is over, the output gap drops drastically (e.g., by more than 3 percent with $\tau = 0.169$). Moreover, there are reversals in output-gap dynamics that were first reported in Carlstrom et al. (2015). Note that under this parameterization, τs are very small (to bound the initial and post-FG inflation, the central bank chooses $\tau = 0.139$). If τ is extended to the size comparable to the previous model, the effects of FG on both output and inflation are even more dramatic). For example, under $\tau = 0.8$, the original Cochrane (2017) model predicts that inflation goes down by 2 percent (this case is not reported).

Future-inflation model. I next perform the same experiments in the future-inflation model; see Figure A4.

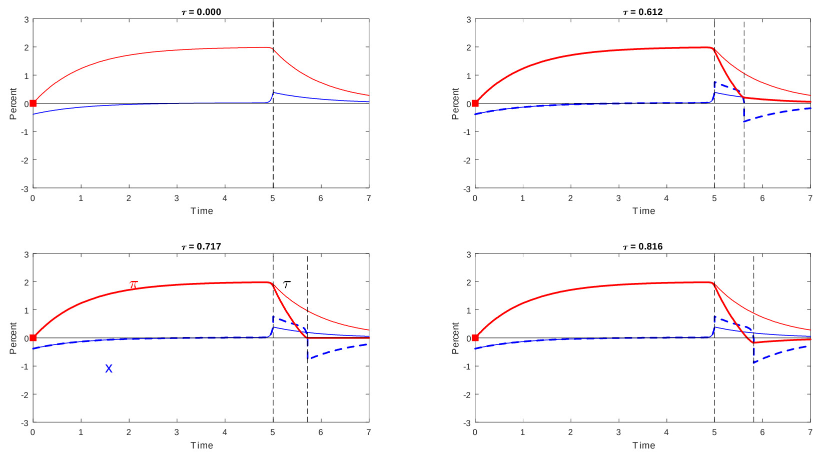

Figure A4. The future-inflation model: a no-jump equilibrium ($\pi_0 = 0$) under alternative parameterization ($k = 5$, $\sigma = 5$).

In this model, a decline in the output gap is much smaller than in the current-inflation model (in spite of the fact that τ is much more sizable now). Moreover, there is no drastic decrease in inflation after the FG period, as in the current-inflation model under a comparable τ.

In sum, although there are no qualitative differences in the effectiveness of the FG policy in the two models, this policy effect in the future-inflation model is far less pronounced. This implication is important because one of the equilibrium selection criteria proposed in Cochrane (2017) was to select an equilibrium with less dramatic predictions ("pick equilibria in which small frictions have small effects"). The future-inflation model will pass such a selection test, while the current-inflation model may not.

On the other hand, if one uses an empirical criterion for equilibrium selection, then the no-jump equilibrium in both models will not survive. In particular, the models will not fit the data about the recent economic crisis when the economy suffered from a deep recession with deflation at the zero lower bound. Both models predict about 2 percent inflation and a fairly small output drop in the beginning of the liquidity-trap period.

The standard equilibrium ($\pi_{T+\tau} = 0$) under $k = 1$, $\sigma = 1$

In this section, I provide the results for the standard equilibrium under benchmark parameterization $k = 1$, $\sigma = 1$.

Current-inflation model. The standard equilibrium is the one that reaches a zero level of inflation at the end of the FG period ($\pi_{T+\tau} = 0$); see Figure A5.

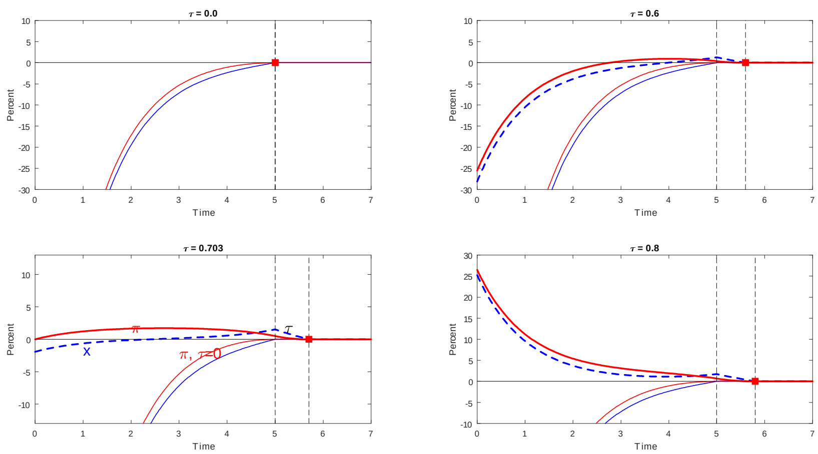

Figure A5. The current-inflation model: a standard equilibrium ($\pi_{T+\tau} = 0$) under benchmark parameterization ($k = 1$, $\sigma = 1$).

The current-inflation model studied in Cochrane (2017) predicts huge effects of liquidity trap on output and inflation. Under the FG policy, the output gap may initially experience either a substantial decrease or a tiny decrease or a sizable increase, depending on the choice of τ. Inflation follows a similar pattern as the output gap. Thus, the central bank can choose the right size of the FG period τ (in our experiments, $\tau = 0.8$) which would stimulate the economy. One of the equilibrium selection criteria proposed in Cochrane (2017) is to use the property that "equilibria should not explode backward". The equilibria displayed in Figure A5 mostly do not satisfy this criterion (except for the FG case with $\tau = 0.703$).

Future-inflation model. On the contrary to the previous model, the changes in output and inflation are far less sizable in the future-inflation model; see Figure A6.

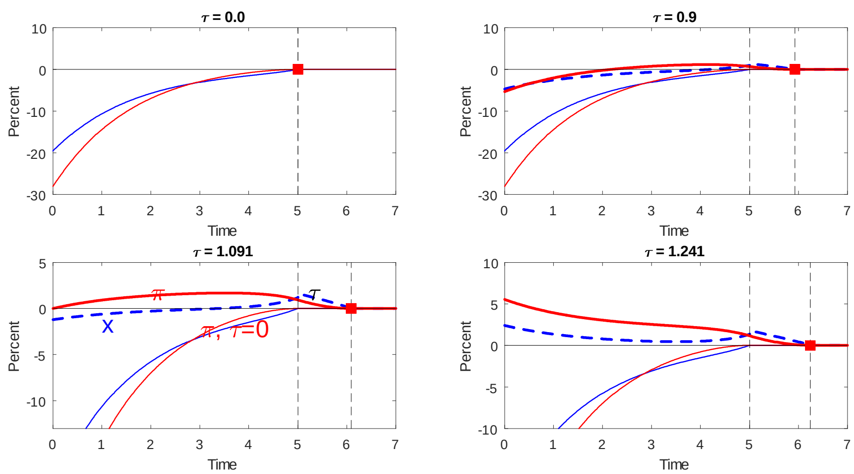

Figure A6. The future-inflation model: a standard equilibrium ($\pi_{T+\tau} = 0$) under benchmark parameterization ($k = 1$, $\sigma = 1$).

For example, instead of hundreds percent of an output reduction due to liquidity trap, I observe here only a 20 percent reduction. Moreover, the effect of the FG policy might be qualitatively different: it is negative, with a small hump, directly during the FG period, for $\tau = 1.091$ and $\tau = 0.9$, but it is slightly positive under $\tau = 1.241$.

Therefore, there are more discrepancies in the implications of the current- and future-inflation models for the standard equilibrium than for the no-jump equilibrium. Moreover, this case demonstrates that the results about the effectiveness of FG are sensitive to the model's specification: a small modification in the model may dramatically change its implications. That is, FG is essentially ineffective in the future-inflation model, while it produces backward explosion in the current-inflation model.