Dyn Games Appl (2016) 6:243–261
DOI 10.1007/s13235-015-0177-8

Ruling Out Multiplicity of Smooth Equilibria in Dynamic Games: A Hyperbolic Discounting Example

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Published online: 23 December 2015

Abstract The literature that conducts numerical analysis of equilibrium in models with hyperbolic (quasi-geometric) discounting reports difficulties in achieving convergence. Surprisingly, numerical methods fail to converge even in a simple, deterministic optimal growth problem that has a well-behaved, smooth closed-form solution. We argue that the reason for nonconvergence is that the generalized Euler equation has a continuum of smooth solutions, each of which is characterized by a different integration constant. We propose two types of restrictions that can rule out the multiplicity: boundary conditions and shape restrictions on equilibrium policy functions. With these additional restrictions, the studied numerical methods deliver a unique smooth solution for both the deterministic and stochastic problems in a wide range of the model's parameters.

Keywords Hyperbolic discounting · Quasi-geometric discounting · Time inconsistency · Markov perfect equilibrium · Markov games · Turnpike theorem · Neoclassical growth model · Endogenous gridpoints · Envelope condition

JEL Classification C73 · D11 · D80 · D90 · E21 · H63 · P16

1 Introduction

Quasi-geometric (hyperbolic) discounting is a form of time inconsistency in preferences when the discount factor, applied between today and tomorrow, differs from the one applied to any other date further in the future. The first studies on quasi-geometric discounting date

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back to Strotz [65], Pollak [59], and Phelps and Pollak [58], although interest to this topic has recently been revived after the work of Laibson [34].[1]

Pollak [59] and Peleg and Yaari [57] observe that models with time inconsistency can be viewed as dynamic games in which the agent in each different period is interpreted as a different agent—*self*. The current self plays a dynamic game against her future selves. The relevant solution concept in such a strategic environment is Nash equilibrium. In particular, the mainstream of the related literature focuses on stationary Markov Nash equilibrium (see, e.g., [24,30,34,34]) and we follow this literature in the present paper.[2]

The problem of solving dynamic models with quasi-geometric agents is known to be very difficult. Sufficient conditions for existence and uniqueness in a general setting with quasi-geometric discounting are actually not well understood. Further, sufficient conditions for a sharp characterization of Markov equilibrium are known only under very strong assumptions. Moreover, the literature that aims to solve numerically the models with quasi-geometric discounting reports difficulties in attaining convergence. Surprisingly, nonconvergence is documented even for a special case of the standard growth model in which a unique smooth closed-form solution is derived analytically and is known to exist; even when the degree of time inconsistency is very small and even if iteration starts in a close neighborhood of the given closed-form solution; see Krusell and Smith [29] and Maliar and Maliar [39,41].

The present paper has two goals: First, we try to gain understanding into why numerical methods iterating on a generalized Euler equation may fail to produce a smooth equilibrium; and second, we aim to provide recommendations on how a unique smooth equilibrium can be reliably constructed.

To investigate the reason for non-convergence of generalized Euler equation methods, we study the implications of the envelope condition in the presence of quasi-geometric discounting. Our analysis builds on a recently developed envelope condition framework in Maliar and Maliar [45].[3] We show that the envelope condition has a continuum of smooth solutions, unlike the usual envelope condition that has a unique smooth solution. As a result, in the presence of quasi-geometric discounting, the Euler equation is a differential equation that contains both a policy function and its derivative and its solution depends on an integration constant. We argue that the transversality condition is not sufficient to discriminate among the multiple solutions. To ensure the uniqueness of the solution, we need to impose additional restrictions that identify the integration constant, for example, to focus on a given steady state or to match a specific boundary condition. The numerical methods used in the previous literature do not impose such additional restrictions, however.

We propose two different approaches for ruling out the multiplicity of equilibria. Our first approach is to restrict a numerical approximation to satisfy an equilibrium boundary

[1] The related literature includes Laibson et al. [35], Barro [9], O'Donoghue and Rabin [55], Harris and Laibson [24], Angeletos et al. [3], Krusell and Smith [29–31], Krusell et al. [32], Luttmer and Mariotti [37], Maliar and Maliar [39–44], Judd [25], Sorger [61], Gong et al. [23], Chatterjee and Eyigungor [16], Balbus et al. [6,7], Bernheim et al. [11], among others.

[2] Other methods can be used in the context of models with quasi-geometric discounting. One of them is a "recursive optimization" approach suggested in Strotz [65], and Caplin and Leahy [14]. A possible implementation of this approach is found in the "pseudo-state space/enlarged state space" analysis of Kydland and Prescott [33] and Feng et al. [22]. Recently, in the context of the game theoretic approach, some literature have suggested turning the problem into a stochastic game (e.g., [24], Balbus et al. [6]). Also, in this latter tradition, one could also attempt to apply incentive-constrained dynamic programming methods [60], recursive dual approaches [49,52,56], and [17], or set-value dynamic programming methods proposed in Abreu, Pearce and Stachetti [1] (e.g., Balbus and Wozny [6]).

[3] For convergence properties of the envelope condition method (ECM) and its applications, see Maliar and Maliar [46], and Arellano et al. [4].

condition that a zero initial capital leads to zero consumption and zero investment. With this additional equilibrium restriction, the studied numerical methods can pin down the appropriate integration constant and can deliver a unique smooth solution for both the deterministic and stochastic problems in a wide range of the model's parameters. The solutions are the same as the closed-form solutions, and the convergence is robust to modifications in the model's and algorithm's parameters, whereas the convergence of unrestricted global Euler equation method are fragile and sensitive to the number of grid points used; see Maliar and Maliar [39,41].

Our second approach is to construct a solution obtained in limit of the solution to a finite horizon economy. This equilibrium selection is motivated by the analysis in Maliar and Maliar [43] who implement the conventional value function iteration (VFI) by "hand" in an example of the model with a known closed-form solution and show that VFI delivers the closed-form solution as a limit of the finite horizon economy.[4] This example suggests that a careful numerical implementation of VFI must produce a unique smooth closed-form solution.[5] By a careful implementation, we mean that a functional form used to parameterize value function must posses the same properties as is imposed when iterating by "hand", namely, it must be continuously differentiable, monotonically increasing and strictly concave. We implement two value function iteration methods, conventional VFI and Carroll's [15] endogenous grid method (EGM), and we find that these methods deliver smooth solutions that are similar to those produced by our restricted generalized Euler equation method under the appropriate shape restrictions.[6]

The rest of the paper is organized as follows. Section 2 formulates the model, derives the optimality conditions and defines the solution concept. Section 3 describes the implications of envelope condition analysis for a generalized Euler equation approaches. Section 4 discusses two approaches for ruling our the multiplicity of equilibria and presents the results of numerical experiments; and finally, Sect. 5 concludes.

2 The Model

We consider a version of the standard neoclassical growth model in which the agent's preferences are time inconsistent because of quasi-geometric discounting.

2.1 The Stochastic Environment

The stochastic environment is standard; see, e.g., Stokey and Lucas with Prescott [64], Santos [62] and Stachurski [63]. Time is discrete, and the horizon is infinite $t = 0, 1, \ldots$ Let $(\Omega, \mathcal{F}, P)$ denote a probability space:

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[4] For the case of the standard geometric discounting, there is a general turnpike theorem that shows that an optimal program of a finite horizon economy asymptotically converges to an optimal program of the corresponding infinite horizon economy under very general assumptions; see Brock and Mirman [13], McKenzie [51], Joshi [26], Majumdar and Zilcha [38], Mitra and Nyarko [54], Becker [10], and Maliar et al. [47]. Turnpike theorems are also known for some dynamic games (see [28] for a survey), but they are not yet established for the economy with quasi-geometric discounting like ours.

[5] In the standard geometric discounting case, there are monotone operators that converge to a limiting stationary solution by iteration on the finite horizon dynamic program; see Coleman [18,19], Mirman et al. [53], Datta et al. [20], and Feng et al. [22].

[6] See also Barillas and Fernandez-Villaverde [8], Maliar and Maliar [45], Fella [21] and White [67] for extensions and applications of EGM.

a) $\Omega = \Pi_{t=0}^{\infty}\Omega_t$ represents a space of sequences $\varepsilon \equiv (\varepsilon_0, \varepsilon_1 \ldots)$ with $\varepsilon_t \in \Omega_t$ for all t. Here, Ω_t is a compact metric space endowed with the Borel σ—field $\mathcal{E}_t$. That is, Ω_t is the set of all possible states of the environment at t and $\varepsilon_t \in \Omega_t$ is the state of the environment at t.
b) $\mathcal{F}$ is the σ—algebra on Ω produced by sets of the form $\Pi_{\tau=0}^{\infty}A_\tau$, where $A_\tau \in \mathcal{E}_\tau$ for all τ.
c) P is the probability measure on $(\Omega, \mathcal{F})$.

$\{\mathcal{F}_t\}$ denotes a filtration on Ω, where $\mathcal{F}_t$ is a sub σ—field of $\mathcal{F}$ induced by a partial history $h_t = (\varepsilon_0, \ldots, \varepsilon_t) \in \Pi_{\tau=0}^{t}\Omega_\tau$ up to period t, i.e., $\mathcal{F}_t$ is generated by sets of the form $\Pi_{\tau=0}^{t}A_\tau$, where $A_\tau \in \mathcal{E}_\tau$ for all $\tau \le t$ and $A_\tau = \Omega_\tau$ for $\tau > t$. In particular, we have that $\mathcal{F}_0$ is the course σ—field $\{0, \Omega\}$ and that $\mathcal{F}_\infty = \mathcal{F}$. If Ω consists of either finite or countable states, ε is called a *discrete state process* or *chain*; otherwise, it is called a *continuous state process*.

2.2 Optimization Problem

We consider a stochastic growth model ran by a planner who solves the following utility maximization problem on each date t:

$$\max_{\{c_\tau, k_{\tau+1}\}_{\tau=t}^{\infty}} \left\{ u(c_t) + E_t \sum_{\tau=t}^{\infty} \beta\delta^{\tau+1-t} u(c_{\tau+1}) \right\} \tag{1}$$

$$\text{s.t. } c_\tau + k_{\tau+1} = (1-d)\,k_\tau + z_\tau f(k_\tau), \tag{2}$$

$$z_{\tau+1} = \varphi(z_\tau, \varepsilon_{\tau+1}), \tag{3}$$

where $c_\tau \ge 0$ and $k_\tau \ge 0$ denote consumption and capital, respectively; initial condition (k_0, z_0) is given; $u : \mathbb{R}_+ \to \mathbb{R}$ and $f : \mathbb{R}_+^2 \to \mathbb{R}_+$ and $\varphi : \mathbb{R}^2 \to \mathbb{R}$ are a utility function, production function and law of motion for an exogenous variable z_t, respectively; ε_{t+1} is i.i.d; $\delta \in (0, 1)$ and $\beta > 0$ are the long- and short-run discount factors, respectively; $d \in [0, 1]$ is the depreciation rate; and $E_t[\cdot]$ is an operator of expectation, conditional on a t-period information set.

We assume that u and f are twice continuously differentiable on $\mathbb{R}_+$, strictly increasing, concave and satisfy the Inada conditions. Moreover, we assume that the objective function in (1) is bounded to ensure the existence of maximum.

The utility weights in (1) decline over time as $\{1, \beta\delta, \beta\delta^2, \ldots\}$, i.e., the utility of period $t+1$ is discounted at a rate $\beta\delta$, and all subsequent utilities are discounted at a rate δ, i.e., the utility weights decline geometrically over time with an exception of the initial period. This discounting is referred to as *quasi-geometric*. In the case of the standard geometric discounting, the agent discounts the utility of all future periods at identical rates, i.e., the utility weights decline geometrically over time $\{1, \delta, \delta^2, \ldots\}$.

2.3 Solution Concepts

Definition 1 (*Feasible program*). A feasible program for the economy (1)–(3) is a pair of adapted (i.e., F_t measurable for all t) processes $\{c_t, k_t\}_{t=0}^{\infty}$ such that given initial condition k_0 and history $h_t = (\varepsilon_0, \varepsilon_1, \ldots)$, they satisfy $c_t \ge 0$, $k_t \ge 0$ and (2) for all t.

We denote by $\Im^\infty$ a set of all feasible programs from given initial capital k_0 and given history $h_t = (\varepsilon_0, \varepsilon_1, \ldots)$.

2.3.1 Optimal Program

We first introduce the concept of solution for the model with geometric discounting.

Definition 2 (*Optimal program with geometric discounting*). A feasible program $\left\{c_t^*, k_t^*\right\}_{t=0}^{\infty} \in \Im^{\infty}$ is called optimal if

$$E_0\left[\sum_{t=0}^{\infty} \delta^t \left\{u\left(c_t^*\right) - u\left(c_t\right)\right\}\right] \geq 0 \tag{4}$$

for every feasible process $\{c_t, k_t\}_{t=0}^{\infty} \in \Im^{\infty}$.

The definition of the optimal program (4) applies only to geometric discounting case $\beta = 1$. In this case, the optimal program $\left\{c_t^*, k_t^*\right\}_{t=0}^{\infty}$, constructed in period $t = 0$, remains optimal in all subsequent periods in the sense that the agent would not change this program in any future period if she could do so.

However, if $\beta \neq 1$, the agent may re-decide on the optimal program as time evolves. If $\beta > 1$ ($\beta < 1$), then the short-run discount factor between t and $t+1$, $\beta\delta$, is higher (lower) than the long-run discount factor δ between any two periods $t+\tau$ and $t+\tau+1$ that are further away in the future $\tau \geq 1$. This leads to time inconsistency in preferences, namely, the relative utility values in any two adjacent periods change depending on whether the agent considers a short run (i.e., next period) or long run (i.e., any periods after the next one). As a result, a sequence $\left\{c_\tau^{*t}, k_{\tau+1}^{*t}\right\}_{\tau=t}^{\infty}$ that is optimal from the point of view of the agent at t may not be the same as the sequence $\left\{c_\tau^{*t'}, k_{\tau+1}^{*t'}\right\}_{\tau=t'}^{\infty}$ that is optimal from the point of view of the agent at $t' > t$. Hence, an agent with time-inconsistent preferences may re-decide on the optimal sequence later on. We assume that the agent is fully aware of her time inconsistency and also, that she cannot commit herself to fulfilling her plans (the problem with commitment is equivalent to the standard geometric discounting case and is straightforward).

2.3.2 Nash Equilibrium

Peleg and Yaari [57] notice that models with time inconsistency can be viewed as dynamic games. An agent at each different date is interpreted as a different agent—*self*. Effectively, a self t decides on consumption and savings only in the current period t because consumption and savings in future periods will be re-decided by future selves. A current (t-period) self plays a dynamic game against her future selves. A relevant solution concept for dynamic games is a Nash equilibrium.

Definition 3 (*Nash equilibrium with quasi-geometric discounting*). A feasible program $\left\{c_\tau^*, k_\tau^*\right\}_{\tau=t}^{\infty} \in \Im^{\infty}$ is called a Nash equilibrium (NE) if for every history h_t and for every feasible program $\{c_\tau, k_\tau\}_{\tau=t}^{\infty} \in \Im^{\infty}$, we have

$$u\left(c_t^*\right) + \beta\delta V_{t+1}^* \geq u\left(c_t\right) + \beta\delta V_{t+1}, \tag{5}$$

where $V_{t+1}^* \equiv E_0\left[\sum_{\tau=t+1}^{\infty} \delta^{\tau-t-1} u\left(c_\tau^*\right)\right]$ and $V_{t+1} \equiv E_0\left[\sum_{\tau=t+1}^{\infty} \delta^{\tau-t} u\left(c_\tau\right)\right]$ are the continuation values for the programs $\left\{c_\tau^*, k_\tau^*\right\}_{\tau=t}^{\infty}$ and $\{c_\tau, k_\tau\}_{\tau=t}^{\infty}$, respectively.

The program $\left\{c_\tau^*, k_\tau^*\right\}_{\tau=t}^{\infty}$ constitutes a NE by definition: if a self τ deviates to any other feasible program $\{c_\tau, k_\tau\}_{\tau=t}^{\infty}$, she will get utility, which is not higher than the one under the program $\left\{c_\tau^*, k_\tau^*\right\}_{\tau=t}^{\infty}$, as (5) implies.

The proposed definition leads to a large set of equilibria, including those built on history dependent strategies, such as a trigger (stick-and-carrot) type of equilibria. In particular, there are many equilibria that are supported by noncredible threats, e.g. a zero-consumption punishment. A novel work of Bernheim et al. [11] provides a complete characterization of a set of all subgame perfect Nash equilibria in a related model: the property of subgame perfection rules out unrealistic equilibria supported by non-credible threats.

2.3.3 Markov Equilibrium

It is common in macroeconomic literature to focus on a more restrictive class of Markov equilibria; see Kocherlakota [27] and Maskine and Tirole [50] for a discussion. The key property of Markov equilibrium is that it is memoryless, i.e., past history is irrelevant for determining the current choices except of the recent past. Moreover, for the infinite horizon economy, it is normally assumed that the equilibrium is stationary, namely, the planner chooses the next-period capital stock k_{t+1} according to a time-invariant policy function, $k_{t+1} = K(k_t, z_t)$.

To characterize a stationary Markov Nash equilibrium in the economy with quasi-geometric discounting (1)–(3), we denote by $W(k_t, z_t)$ the optimal value of the expected discounted utility of the agent whose current state is k_t and z_t, and who from period $t+1$ and on, makes her decisions according to a time-invariant policy function K. A recursive formulation of the model (1)–(3) is

$$W(k, z) = \max_{k'} \left\{u\left((1-d)k + zf(k) - k'\right) + \beta\delta E\left[V\left(k', z'\right) \mid z\right]\right\}, \tag{6}$$

where the value function of future selves $V(k, z)$ is defined by a recursive functional equation

$$V(k, z) = u(1-d)k + zf(k) - k' + \delta E\left\{V\left(k', z'\right) \mid z\right\}, \tag{7}$$

and k, z are given.

Definition 4 (*Stationary Markov Nash equilibrium with quasi-geometric discounting*). A stationary Markov Nash equilibrium (SMNE) is a collection of functions $W : \mathbb{R}_+^2 \to \mathbb{R}$, $V : \mathbb{R}_+^2 \to \mathbb{R}$ and $K : \mathbb{R}_+^2 \to \mathbb{R}_+$ that satisfy (6), (7).

The notion of SMNE eliminates the problem of time inconsistency. This equilibrium is time consistent: the agent chooses decision rules that maximize her expected life-time utility function by explicitly taking into account that her preferences change over time.

2.4 A Smooth Closed-Form Solution

As was first noticed by Krusell and Smith [29], the model (1)–(3) admits a closed-form solution under one parameterization.[7] Namely, assume $u(c) = \ln c$, $f(k) = k^{\alpha}$, with $\alpha \in (0, 1)$ and $d = 1$. Also, assume that the equilibrium is interior and smooth. Then, it is easy to verify that the optimal value and policy functions are given by

[7] Maliar and Maliar [40] shows another example of the model with quasi-geometric discounting that admits a closed-form solution under the assumption of the exponential utility function.

$$V(k,z) = (1-\delta)^{-1}\left(\ln\frac{1-\delta\alpha}{1-\delta\alpha+\beta\delta\alpha} + \frac{\delta\alpha}{1-\delta\alpha}\ln\frac{\beta\delta\alpha}{1-\delta\alpha+\beta\delta\alpha}\right) + \frac{\alpha}{1-\delta\alpha}\ln k + \frac{1}{(1-\delta\rho)(1-\delta\alpha)}\ln z, \tag{8}$$

$$c = \frac{1-\delta\alpha}{1-\delta\alpha+\beta\delta\alpha} z k^{\alpha} \quad \text{and} \quad k' = \frac{\beta\delta\alpha}{1-\delta\alpha+\beta\delta\alpha} z k^{\alpha}. \tag{9}$$

A model with a closed-form solution is convenient for testing numerical methods for the analysis of equilibrium.

3 Why is it so Hard to Find a Smooth Equilibrium?

The numerical analysis of equilibrium in the model with quasi-geometric discounting proved to be problematic. Krusell and Smith [29] study a deterministic version of the model with a closed-form solution (8) and (9), i.e., they assume $z = 1$. They tried to approximate this solution numerically by using the conventional value and policy function iteration based on a discretization of state space. Surprisingly, the conventional numerical methods fail to converge to a known closed-form solution, namely, they either produce cycling or converge to some different solutions with jumps. Krusell and Smith [29] explain their findings by the fact that, in addition to the smooth, closed-form solution, there are infinitely many discontinuous solutions in the form of step functions; see also Krusell and Smith [30,31] for a discussion. Also, discontinuous Markov perfect equilibria are constructed in Bernheim et al. [11].

Subsequent literature that focuses on constructing smooth solutions comes up with contradictory findings. Krusell et al. [32] argue that a perturbation method delivers a unique smooth solution; however, Judd [25] shows that perturbation methods produce multiple solutions if the precision is increased. Maliar and Maliar [39,41] and Judd [25] report that global solution methods such as projection or stochastic simulation methods produce a unique smooth solution. However, Maliar and Maliar [39,41] also find that when the flexibility of approximation function increases (i.e., degree of approximating polynomial or the number of grid points), projection methods also fail to converge. Chatterjee and Eyigungor [16] argue that even if a smooth solution exists, an approximation of such a solution may fail to exist if a domain is truncated. Our goal will be to reconcile these findings and in particular, to gain intuition into why the conventional numerical methods may fail to produce a smooth equilibrium.

3.1 Generalized Euler Equation

The literature on quasi-geometric discounting typically constructs smooth solutions by using a generalized Euler equation. We derive this equation below. If the equilibrium is interior and if V is differentiable, such an equilibrium can be characterized by a first-order condition (FOC) of (6), (7). The FOC with respect to k' is

$$u'(c) = \beta\delta E\left[\frac{\partial V(k',z')}{\partial k'}\right], \tag{10}$$

where the prime on variables is used to denote their future values. The derivative of V with respect to k' is

$$\frac{\partial V(k,z)}{\partial k} = u'(c)\left(1-d+zf'(k)-\frac{\partial K(k,z)}{\partial k}\right) + \frac{\partial K(k,z)}{\partial k}\delta E\left[\frac{\partial V(k',z')}{\partial k'}\right], \tag{11}$$

where $\partial K/\partial k$ is taken out of the expectation because it is known before the shock z' is realized. By substituting (10) into (11), updating and rearranging the terms, we obtain:

$$\beta \frac{\partial V\left(k', z'\right)}{\partial k'} = u'\left(c'\right)\left(\beta\left(1 - d + z' f'\left(k'\right)\right) + (1-\beta)\frac{\partial K\left(k', z'\right)}{\partial k'}\right). \tag{12}$$

With a slight abuse of terminology, we will refer to the above condition as the "envelope condition" although it does not have the property that $\frac{\partial K(k',z')}{\partial k'}$ cancels out, as implied by the envelope theorems. By combining (10) and (12), we obtain a generalized Euler equation

$$u'(c) = \delta E\left\{u'\left(c'\right)\left(\beta\left(1 - d + z' f'\left(k'\right)\right) + (1-\beta)\frac{\partial K\left(k', z'\right)}{\partial k'}\right)\right\}. \tag{13}$$

Note that the agent's consumption-saving decision at time t depends not only on the future return on capital but also on the future marginal propensity to save out of capital, $\frac{\partial K(k',z')}{\partial k'}$. This feature of the model plays a determinant role in the properties of the solution and is not present in the version of the model with standard geometric discounting.

3.2 Multiplicity of Solutions to the Envelope Condition

Maliar and Maliar [39] introduce a recursion for value function iteration, called envelope condition method (ECM), that differs from the standard backward value function iteration (VFI); see Arellano et al. [4] and Maliar and Maliar [46] for a discussion of convergence results of ECM and its further applications. We now show that the ECM analysis has important implications for the model with quasi-geometric discounting (1)–(3).

For expositional convenience, let us consider the deterministic case (i.e., we assume $z_t = 1$ for all t), and let us restrict attention to the model with a closed-form solution, studied in Sect. 2.4. Our goal is to construct the consumption function. By using the budget constraint (2), we re-write the envelope condition (12) in terms of the derivative of the consumption function $c(k)$

$$\beta \frac{\mathrm{d}V(k)}{\mathrm{d}k} = u'(c)\left(zf'(k) - (1-\beta)\frac{\mathrm{d}c(k)}{\mathrm{d}k}\right). \tag{14}$$

By rearranging the terms, we obtain the following differential equation

$$\frac{\beta c}{1-\beta}\frac{\mathrm{d}V(k)}{\mathrm{d}k} + \frac{\mathrm{d}c}{\mathrm{d}k} = \frac{\alpha}{1-\beta}k^{\alpha-1}. \tag{15}$$

We first solve the homogeneous differential equation $\frac{\beta c}{1-\beta}\frac{\mathrm{d}V(k)}{\mathrm{d}k} + \frac{\mathrm{d}c}{\mathrm{d}k} = 0$ which gives us

$$c = E(k)\exp\left(-\frac{\beta V(k)}{1-\beta}\right), \tag{16}$$

where $E(k)$ is an unknown function of k. Then, by substituting (16) into (15), we obtain an equation that identifies $E(k)$

$$E'(k) = \frac{\alpha}{(1-\beta)}k^{\alpha-1}\exp\left(\frac{\beta V(k)}{1-\beta}\right). \tag{17}$$

By integrating $E(k)$ and by substituting it into (16), we receive the following expression for consumption

$$c = \frac{\alpha}{(1-\beta)}\exp\left(-\frac{\beta V(k)}{1-\beta}\right)\int^k x^{\alpha-1}\exp\left(\frac{\beta V(x)}{1-\beta}\right)\mathrm{d}x + D\exp\left(-\frac{\beta V(k)}{1-\beta}\right), \tag{18}$$

where D is a finite constant. Thus, the resulting consumption function depends on an arbitrary constant D and is not uniquely defined, like in the standard geometric discounting case when c can be directly expressed from $\beta \frac{\mathrm{d}V(k)}{\mathrm{d}k} = u'(c)\, zf'(k)$. The multiplicity of solutions to the envelope condition arises even if we insert the true closed-form solution in (18), as the following example shows.

Example (*A quasi-geometric discounting model with a closed-form solution*). Assume $\beta \neq 1$ and substitute the exact value function (8) into (18) to get

$$c = \frac{\alpha}{(1-\beta)} \exp\left(-\frac{\beta\alpha}{(1-\beta)(1-\delta\alpha)} \ln(k)\right) \int^{k} x^{\alpha-1} \exp\left(\frac{\beta\alpha}{(1-\beta)(1-\delta\alpha)} \ln(x)\right) \mathrm{d}x + D \exp\left(-\frac{\beta\alpha}{(1-\beta)(1-\delta\alpha)} \ln(k)\right).$$

By simplifying this expression, we get

$$c = \frac{1-\delta\alpha}{1-\delta\alpha+\beta\delta\alpha} k^{\alpha} + D k^{\frac{-\beta\alpha}{(1-\beta)(1-\delta\alpha)}}. \tag{19}$$

Comparing to the closed-form solution in (9), we can see that the consumption function has an additional term, and it coincides with the closed-form solution only one specific integration constant $D = 0$.

3.3 Generalized Euler Equation in the Integral Form

The analysis of Sect. 3.2 leads us to a different but equivalent representation of the generalized Euler equation. Specifically, FOC (10) for the deterministic model with a closed-form solution is given by

$$\frac{1}{c} = \beta\delta \left[\frac{\mathrm{d}V\left(k'\right)}{\mathrm{d}k'}\right]. \tag{20}$$

Updating equation (18), we can rewrite the envelope condition (12) in the integral form for the next period

$$c' = \frac{\alpha}{(1-\beta)} \exp\left(-\frac{\beta V\left(k'\right)}{1-\beta}\right) \int^{k'} x^{\alpha-1} \exp\left(\frac{\beta V(x)}{1-\beta}\right) \mathrm{d}x + D \exp\left(-\frac{\beta V\left(k'\right)}{1-\beta}\right). \tag{21}$$

The conventional generalized Euler equation (13) is obtained by combining FOC (10) and envelope condition (12) to eliminate the unknown derivative of value function. This is not possible to do with our integral form of the envelope condition (21) because it contains value function $V\left(k'\right)$ and not its derivative. Therefore, we need to solve these two equations jointly with respect to both c and V. The advantage of our representation is that we do not have the derivative of the policy function that is present in the generalized Euler equation (13). The fact that the system of Eqs. (20) and (21) depends on an integration constant D indicates that our multiplicity results obtained for the envelope condition carry over to the generalized Euler equation class of methods.

3.4 Multiplicity of Smooth Solutions is a Generic Property

Multiplicity of smooth solutions satisfying the generalized Euler equation seems to be a generic property of this class of models. Indeed, the envelope condition (12) and similarly, the generalized Euler equation (13) contain both a consumption function c and its derivative

$\frac{\mathrm{d}c}{\mathrm{d}k}$. The solution to this differential equation depends on an integration constant. This is true not only for our test model with a closed-form solution but also for more general versions of the model, as well as for other similar dynamic games in which Euler equations contain both a policy function and its derivative.

In the standard geometric discounting model, the sufficiency condition for maximization is the transversality condition. This condition is ensured by focusing on the solution that converges asymptotically to the steady state. However, the steady state is not well defined in the problems with quasi-geometric discounting because it depends on an unknown derivative $\frac{\mathrm{d}K}{\partial k'}$. Specifically, evaluating the generalized Euler equation (13) in a possible steady state yields

$$1 = \delta\left(\beta\left(1 - d + zf'\left(\overline{k}\right)\right) + (1-\beta)\,\frac{\partial K\left(\overline{k}, \overline{z}\right)}{\partial \overline{k}}\right), \tag{22}$$

where notations with bars denote steady-state values of the corresponding variables. Unless $\frac{\partial K\left(\overline{k},\overline{z}\right)}{\partial \overline{k}}$ is known, it is not possible to compute $\overline{k}$ from (22). In particular, if the model admits multiple steady state satisfying (22), all such steady states will be consistent with the transversality condition by definition. Moreover, Maliar and Maliar [43] show that any potential steady state $\overline{k} \in \left[\left(f'\right)^{-1}\left(\frac{1-(1-d)\beta\delta}{\overline{z}\beta\delta}\right),\ \left(f'\right)^{-1}\left(\frac{1-\delta(1-d\beta)}{\overline{z}\beta\delta}\right)\right]$ is consistent with the property of saddle path stability, similarly to the steady state of the standard growth model with geometric discounting.

4 Ruling Out the Multiplicity of Equilibria

An important practical question is: "Is it possible to discriminate among multiple solutions of type (18)?" We present two approaches that can help to rule out the multiplicity of equilibria: One is to impose some equilibrium boundary conditions on an approximate solution, and the other is to impose the equilibrium shape restrictions such as differentiability, monotonicity and concavity; these two approaches are described in Sects. 4.1 and 4.2, respectively.

4.1 Imposing Boundary Conditions

Consider the equilibrium boundary condition when the capital stock is zero $k = 0$. If the production function is Cobb-Douglas $f(k) = k^{\alpha}$ with $\alpha \in (0, 1)$, the budget constraint (2) implies that if $k = 0$, than $K(0, z) = 0$ and $C(0, z) = 0$ and hence, the constant in the envelope condition (21) must be equal to zero, i.e. $D = 0$.

Below, we argue that a class of global polynomial functions is convenient for imposing such boundary conditions. Namely, assume that the capital function $K(k)$ is parameterized by an ordinary polynomial function $\widehat{K}(k, b)$, i.e.

$$K(k) \approx \widehat{K}(k; b) = b_0 + b_1 k + b_2 k^2 + b_3 k^3 + b_4 k^4 + b_5 k^5 + \ldots + b_n k^n, \tag{23}$$

where $b \equiv (b_0, b_1, \ldots, b_n)$. Then, the boundary condition $D = 0$ implies $\widehat{K}(0; b) = 0$ which means approximation (23) must be constructed under the additional restriction $b_0 = 0$. As an alternative, it is also possible to pin down the integration constant by focusing on a specific steady state.

4.1.1 Numerical Iteration on a Generalized Euler Equation

We perform numerical analysis of equilibrium in the model with quasi-geometric discounting (1)–(3) using the Euler equation algorithm described in Maliar and Maliar [39,41]. We restrict the domain of capital to the interval $[k_{\min}, k_{\max}] = \left[0.5\overline{k}^*, 2\overline{k}^*\right]$, where $\overline{k}^*$ is the steady-state value of capital in the model with standard geometric discounting. We consider an equally spaced grid of N points.

For all numerical experiments, we assume a constant relative risk aversion (CRRA) utility function, $u(c) = \frac{c^{1-\gamma}-1}{1-\gamma}$, where $\gamma > 0$, and a Cobb–Douglas production function, $f(k) = k^{\alpha}$; we fix $\alpha = 0.36$ and $\delta = 0.95$. If $\gamma = 1$ and $d = 1$, we obtain the model with the closed-form solution.

In the stochastic case, we parameterize the process for productivity levels by $\rho = 0.95$ and $\sigma = 0.01$. We approximate the autoregressive process by a Markov chain with seven states, $Z \equiv \left\{0, \pm\frac{5\sigma}{3}, \pm\frac{5\sigma}{2}, \pm 5\sigma\right\}$ and compute the corresponding transition probabilities $\pi\left(z'|z\right)$, as in Tauchen [66]. Our solution algorithm is described below.

Algorithm 1. Generalized Euler-equation iteration

Given $K(k, z; b)$, for each point (k, z), define the following recursion:

i). Compute $c = \left\{\delta \sum_{z' \in Z} \left[\frac{\beta\left(1-d+z'\alpha K(k,z)^{\alpha-1}\right)+(1-\beta)\frac{\partial K(K(k,z),z')}{\partial K(k,z)}}{((1-d)K(k,z)+z'K(k,z)^{\alpha}-K(K(k,z),z'))^{\gamma}}\right]\pi\left(z'|z\right)\right\}^{-1/\gamma}$.

ii). Find $\widehat{k'} = (1-\delta)k + zf(k) - c$.

iii). Find $\widehat{K}(k, z; b)$ that fits $\widehat{k'}$ on the grid (k, z).

Iterate on i)-iii) until convergence $\widehat{K} = K$.

Algorithm 1 implements fixed point iteration on capital policy function. The advantage of this algorithm is that it relies on direct calculations and does not use either maximization or equation solving routines. The shortcoming is that the convergence of fixed point iteration is not guaranteed; see Maliar and Maliar [46] for a discussion. To enhance the convergence properties of the studied method, we begin iteration sufficiently close to the exact solution and we use partial updating of policy function along iterations $\lambda\widehat{K} + (1-\lambda)K$, where $\lambda = 0.01$, i.e., we update the solution by just 1 % on each iteration.

4.1.2 Cycling, Nonconvergence and Multiple Solutions in the Presence of Quasi-Geometric Discounting

Maliar and Maliar [39,41] applied Algorithm 1 to similar problems with quasi-geometric discounting by evaluating policy functions outside the grid using cubic spline interpolation. These papers find that iteration on quasi-geometric Euler equation (13) using Algorithm 1 delivers the closed-form solution (8), (9) when an approximating function was relatively rigid and inflexible, but it starts cycling and deviates from the closed-form solution when a more flexible functional form is used. This was true even if the degree of time inconsistency is very small and even if iteration begins arbitrary closed to a known closed-form solution. In Fig. 1, we illustrate an example of the constructed solutions depending on the number of grid point used for the model with the closed-form solution under $\beta = 0.5$.

In the very first panel (upper-left), we use a relatively inflexible numerical approximation with just 30 points. In this case, the approximate solution is visually indistinguishable from the closed-form solution. When we gradually increase the number of grid points from 30

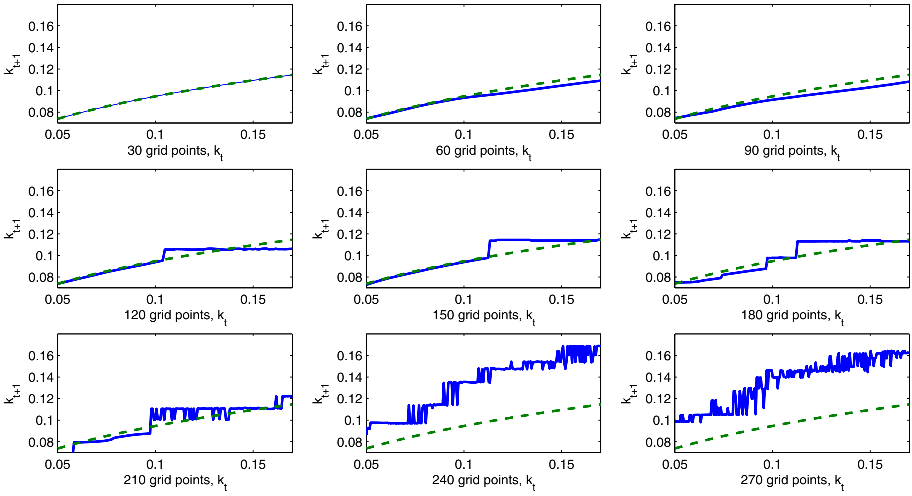

Fig. 1 Cycling, nonconvergence and multiple solutions

to 270, we observe that the numerical method produce solutions that visibly differ from the closed-form solution. In the bottom three panels (210, 240 and 270 grid points), the algorithm failed to converge. Our results on multiplicity in Sect. 3 help us to shed some light on why numerical procedures based on a generalized Euler equation may produce this type of behavior. Namely, unrestricted cubic interpolation does not allow us to identify specific integration constant in the solution (18) and the numerical method randomly selects one of many possible solutions.

4.1.3 Deterministic Model with a Boundary Equilibrium Condition

We then solved a deterministic version of the model by using global polynomial approximation (23) under the additional restriction $b_0 = 0$ that implies $D = 0$ in the envelope condition (18). We find that the generalized Euler equation method systematically converges to a closed-form solution under a wide range of β, including large degrees of time inconsistency such as $\beta = 0.5$ and $\beta = 1.5$. We plot the capital functions under three selected values of β in Fig. 2.

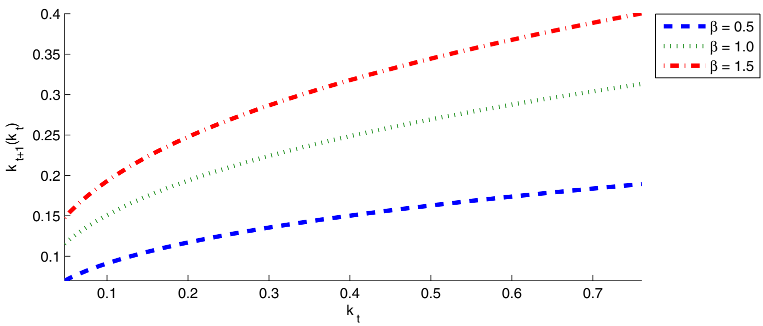

Fig. 2 Solution to the deterministic model

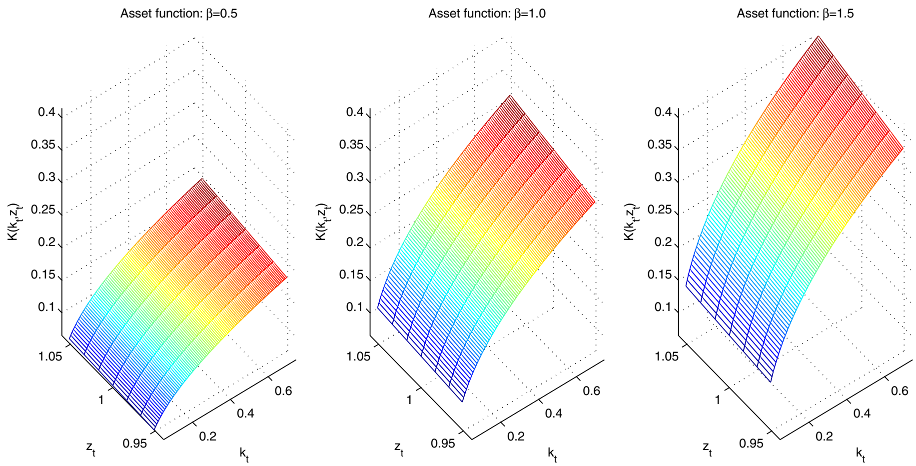

Fig. 3 Solution to the stochastic model

The noteworthy finding in the figures is that the solutions under all three values of β are very similar. The main difference is that an agent with $\beta > 1$ ($\beta < 1$) holds more (less) capital than the one with $\beta = 1$, i.e., the short-run patient (impatient) agent tends to over-save (under-save) relative to the one with $\beta = 1$. The solutions we obtain are essentially identical to those constructed by Maliar and Maliar [39,41] using a grid-based numerical method iterating on a generalized Euler equation. However, Maliar and Maliar [39,41] report the problem of nonconvergence when the number of grid points increases, while we do not observe this problem even with a very large number of grid points such as 10,000.

We run sensitivity experiments. For the model with a closed-form solution. Our generalized Euler equation method delivers a closed-form solution for such a large range of the degrees of quasi-geometric discounting as $\beta \in [0.05, 3]$. We also solved the model with partial depreciation of capital $d = 0.025$ and the degrees of risk aversion in the range $\gamma \in \left[\frac{1}{5}, 10\right]$, and we find that the algorithm is systematically converging to a unique smooth solution.

4.1.4 Stochastic Model with Boundary Equilibrium Conditions

In the stochastic model, the performance of the studied method was also successful: the algorithm was again able to converge to a closed-form solution for both small and large values of β. We plot the asset (capital) functions for $\beta = 0.5$, $\beta = 1$ and $\beta = 1.5$ in Fig. 3.

Similar to the deterministic case, short-run patient agents save more than short-run impatient agents. This is true for any level of exogenous productivity level. We do not observe cycling when the number of grid points increases.

4.2 Imposing Shape Restrictions

In this section, we explore another strategy for ruling out the multiplicity of equilibrium, namely, we restrict attention to the limit of the finite horizon version of the model. We show that the finite horizon problem converges to the infinite horizon economy as $T \to \infty$ under the parameterization that leads to a closed-form solution. This result suggests that a smooth SMNE can be reliably constructed by using iteration on Bellman equation, provided that we impose appropriate restrictions on the interpolating function such as smoothness, monotonicity, differentiability and concavity.[8]

4.2.1 Finite Horizon Economy

In this section, we consider a finite horizon version of the economy (1)–(3) with a terminal period T:

$$\max_{\{c_\tau, k_{\tau+1}\}_{\tau=t}^{\infty}} \left\{ u(c_t) + E_t \sum_{\tau=t}^{T} \beta \delta^{\tau+1-t} u(c_{\tau+1}) \right\} \tag{24}$$

$$\text{s.t. (2), (3),} \tag{25}$$

where initial condition (k_0, z_0) is given. The solution concepts developed for the infinite horizon economy (1)–(3) apply to the finite horizon case as well. In particular, the parallel definitions of the optimal program (4) and Nash equilibrium (5) follow by replacing the infinite horizon ∞ with the finite horizon T. The definition of Markov equilibria for a finite horizon economy is also similar but the property of stationarity is not imposed in the finite horizon case. Here, the optimal value and policy are time dependent and are characterized by backward induction. The standard value iterative method that constructs a solution by backward induction is known as time-iteration.

Algorithm 2. Value function iteration

Set some V_{T+1} and compute backward $\{V_t, W_t\}_{t=T,\ldots,0}$ using:

$$W_t(k_t, z_t) = \max_{k_t} \left\{ u((1-d)k_t + z_t f(k_t) - k_{t+1}) + \beta\delta E_t \left[V_{t+1}(k_{t+1}, z_{t+1}) \right] \right\},$$

$$V_t(k_t, z_t) = u((1-d)k_t + z_t f(k_t) - k_{t+1}) + \delta E_t \left[V_{t+1}(k_{t+1}, z_{t+1}) \right].$$

Definition 5 (*Markov perfect equilibria with quasi-geometric discounting*). A Markov perfect equilibrium (MPE) in the finite horizon economy is a collection of functions $W_t : \mathbb{R}_+^2 \to \mathbb{R}$, $V_t : \mathbb{R}_+^2 \to \mathbb{R}$ and $K_t : \mathbb{R}_+^2 \to \mathbb{R}_+$, $t = 0, \ldots, T$ constructed by Algorithm 2.

In the standard geometric discounting case, it is known that the finite horizon economy (24), (25) asymptotically converges to the infinite horizon economy (1)–(3) as $T \to \infty$. This kind of convergence results is referred to as *turnpike theorems* ; see, e.g., Brock and Mirman [13], McKenzie [51], Majumdar and Zilcha [38], Mitra and Nyarko [54], Joshi [26], Becker [10] and Maliar et al. [47].

There is an example suggesting that turnpike-style results can hold for some quasi-geometric discounting problems. Namely, Maliar and Maliar ([43], Appendix A) apply Algorithm 2 to iterate "by hand" on the Bellman equation of the finite horizon model using Algorithm 2 and show that the limiting value and policy functions converge to the closed-form solution (8), (9) of the infinite horizon economy. (The above result is an extension of the analysis of Manuelli and Sargent [48] to the quasi-geometric discounting case). This indicates that, at least for a special case of the model with the closed-form solution, the finite horizon problem converges in the limit to the infinite horizon problem.

[8] Balbus, Reffett and Wozny [6,7] suggest a different value-based recursion for the model with qusi-geometric discounting, which under appropriate assumptions delivers a unique SMNE equilibrium using a simple successive approximation scheme.

4.2.2 Value Function Iteration Methods

We implement two methods that perform backward value function iteration, namely, the conventional VFI and endogenous grid method (EGM) of Carroll [15]. Both methods guess value function at $t+1$ and use the Bellman equation to compute value function at t. FOC (10), combined with budget constraint (2), becomes

$$u'(c) = \beta\delta E\left[V_1\left((1-\delta)k + zf(k) - c, z'\right)\right]. \tag{26}$$

Conventional VFI finds consumption c by calculating a solution to FOC (26).

Algorithm 3. Conventional VFI

Given V, for each point (k, z), define the following recursion:
- i). Solve for c satisfying $u'(c) = \beta\delta E\left[V_1\left((1-\delta)k + zf(k) - c, z'\right)\right]$.
- ii). Find $k' = (1-\delta)k + zf(k) - c$.
- iii). Find $\widehat{V}(k, z) = u(c) + \beta E\left[V\left(k', z'\right)\right]$.

Iterate on i)-iii) until convergence $\widehat{V} = V$.

Conventional VFI is expensive because Step i) requires us to numerically find a root to (26) for each (k, z) by interpolating V_1 to new values $\left(k', z'\right)$ and by approximating conditional expectation —this all must be done inside an iterative cycle; see Aruoba et al. [5] for an example of cost assessment of conventional VFI. (Alternatively, we can find k' by maximizing the right side of Bellman equation (8) directly without using FOCs, however, this is also expensive).

Carroll [15] proposes a way to reduce the cost of conventional VFI. The EGM method of Carroll [15] exploits the fact that it is easier to solve (26) with respect to c given $\left(k', z\right)$ than to solve it with respect to c given (k, z). EGM constructs a grid on $\left(k', z\right)$ by fixing the future endogenous state variable k' and by treating the current endogenous state variable k as unknown. Since k' is fixed, EGM computes $E\left[V_1\left(k', z'\right)\right]$ up-front and thus can avoid costly interpolation and approximation of expectation in a rootfinding procedure.

Algorithm 4. EGM of Carroll [15]

Given V, for each point $\left(k', z\right)$, define the following recursion:
- i). Find $c = u'^{-1}\left\{\beta\delta E\left[V_1\left(k', z'\right)\right]\right\}$.
- ii). Solve for k satisfying $k' = (1-\delta)k + zf(k) - c$.
- iii). Find $\widehat{V}(k, z) = u(c) + \beta E\left[V\left(k', z'\right)\right]$.

Iterate on i)-iii) until convergence $\widehat{V} = V$.

In Step ii) of EGM, we still need to find k numerically. However, for the studied model, Carroll [15] shows a change of variables that makes it possible to avoid finding k numerically on each iteration (except of the very last iteration).

4.2.3 Numerical Experiments with Shape Restrictions

Why do we expect VFI and EGM to work? This is because they both use the same backward iteration by "hand" as in Maliar and Maliar [43] and thus, they must lead to the same limit provided that we impose the same restrictions on the interpolating function as those imposed in the iteration by "hand", namely, monotonicity, differentiability and concavity. These restrictions are not enforced in the value iteration method of Krusell and Smith

[29–31] based on discretization of state space and such methods produce cycling among a large set of smooth and nonsmooth equilibria.

We run a number of numerical experiments, and we indeed observed under appropriate shape restrictions on the approximating functions (i.e., monotonicity, differentiability and concavity), the studied value function iteration methods converge systematically and produce solutions that are similar to those reported in Sect. 3 (to save on space, these experiments are not reported).

Moreover, we find that imposing such shape restrictions on the approximating functions enhances the convergence properties of numerical methods iterating on the generalized Euler equation. In particular, Algorithm 1 does not encounter the problems of cycling and nonconvergence reported by Maliar and Maliar [39,41] when the number of grid points becomes large. We were able to systematically construct a unique smooth solutions both in the version of the model with the closed-form solution and more general versions of the model with a very large number of grid point.

4.2.4 Discussion

Our turnpike analysis is limited to one particular example of the model with closed-form solution. It is not known whether or not parallel results hold for more general version of the model with quasi-geometric discounting (1)–(3). The construction of turnpike results for dynamic games like this is complicated by the facts that, first, the existence and uniqueness of SMNE in infinite horizon economy is not guaranteed and second, the uniqueness of MPE in finite horizon economy is also not guaranteed, see Bernheim and Ray [12] and Leininger [36] for a discussion. For a survey of the existing turnpike results for dynamic games, see Kolokoltsov and Yang [28].

Furthermore, it is also not known whether or not the standard value and policy function iteration will systematically converge to the infinite horizon solution in the limit even if such limit happens to exist. For the standard geometric discounting model, there are value function and Euler equation monotone operators that are known to uniformly converge to a limiting stationary solution by simple iteration on the finite horizon dynamic program; see Coleman [18] and Mirman et al. [53] and also, see Coleman [19], Datta et al. [20], and Feng et al. [22] for related convergence results. However, in the presence of derivatives of policy functions in the Euler equation, the pointwise decreasing consumption (and associated pointwise increasing investment) along turnpikes does not hold; hence, the structure of the turnpike properties relative to Coleman's [18] policy iteration method fails in general. This is because partial orders, where monotonicity of operators can be preserved for solving functional equations that involve derivative properties of the space of functions, are difficult to construct. This is exactly the problem noted in the seminal paper by Amann [2] on using partial ordering methods to solve differential equations, namely, the partial orders typically do not reflect gradient properties of policy functions. In other words, monotonicity is hard to attain when iterating on functional equations that contain both policy functions and their derivatives.

Finally, for the optimal growth model with the closed-form solution, the appropriate shape restrictions are known because a closed-form solution is available. Some properties of the solutions can be established for more general versions of the model, for example, Maliar and Maliar [42] show monotonicity and strict concavity of the asset and consumption functions in a related model with quasi-geometric discounting. However, in more complex models, the shape of equilibrium functions may be hard to establish, so it might be unclear what restrictions must be imposed on the equilibrium functions.

5 Conclusion

The previous literature has encountered difficulties in constructing numerical solutions to an optimal growth model with quasi-geometric discounting because numerical methods cycle among a large set of equilibria. To reduce the set of equilibria, some papers suggested to focus on a smooth solution satisfying the generalized Euler equation. However, our envelope condition analysis indicates that there are multiple smooth solutions satisfying generalized Euler equation as well. Our argument is both general and intuitive: Since a generalized Euler equation contains both a policy functions and its derivative, the solution to the resulting differential equation depends on an integration constant. This is the source of multiplicity of smooth equilibria.

We propose two ways of ruling out the multiplicity of equilibrium. First, we argue that it is possible to pin down a specific integration constant by imposing some additional restriction on the constructed solutions such as a given boundary condition, a specific steady state or similar. Second, we find that imposing additional shape restrictions on the interpolant such as monotonicity, differentiability and concavity can be sufficient for making numerical methods to systematically converge to the closed-form solution in our examples.

While our analysis is limited to a model with quasi-geometric discounting, its implications carry over to a variety of dynamic strategic contexts in which time inconsistency is involved, including government policy problems, monopolistic competition, etc. It would be of interest to explore limiting properties of finite horizon versions of such problems and to establish general turnpike-style results on their asymptotic convergence.

Acknowledgments Lilia Maliar and Serguei Maliar acknowledge support from the Hoover Institution and Department of Economics at Stanford University, University of Alicante, Santa Clara University and MECD Grant ECO2012-36719. We thank the editors Edward Prescott and Kevin Reffett for many useful comments and suggestions.

References

1. Abreu D, Pearce D, Stachetti E (1986) Optimal cartel equilibria with imperfect monitoring. J Econ Theory 39:251–269
2. Amann H (1976) Fixed point equations and nonlinear eigenvalue problems in ordered banach spaces. SIAM Rev 18(4):620–709
3. Angeletos M, Laibson D, Repetto A, Tobacman J, Weinberg S (2001) The hyperbolic consumption model: calibration, simulation, and empirical evaluation. J Econ Perspect 15(3):47–68
4. Arellano C, Maliar L, Maliar S, Tsyrennikov V (2014) Envelope condition method with an application to default risk models (Manuscript)
5. Aruoba SB, Fernández-Villaverde J, Rubio-Ramí rez J (2006) Comparing solution methods for dynamic equilibrium economies. J Econ Dyn Control 30:2477–2508
6. Balbus Ł, Reffet K, Woźny Ł (2015) Time consistent Markov policies in dynamic economies with quasi-hyperbolic consumers. Int J Game Theory 44(1):83–112
7. Balbus L, Reffet K, Wozny L (2012) Existence and uniqueness of time-consistent Markov policies for quasi-hyperbolic consumers under uncertainty (Manuscript)
8. Barillas F, Fernandez-Villaverde J (2007) A generalization of the endogenous grid method. J Econ Dyn Control 31(8):2698–2712
9. Barro R (1999) Ramsey meets Laibson in the neoclassical growth model. Q J Econ 114(4):1125–1152
10. Becker R (2012) Optimal growth with heterogeneous agents and the twisted turnpike: an example. Int J Econ Theory 8(1):24–47
11. Bernheim D, Ray D, Yeltekin S (2015) Poverty and self-control. Econometrica (forthcoming)
12. Bernheim D, Ray D (1986) On the existence of Markov-consistent plans under production uncertainty. Rev Econ Stud 53(5):877–882

13. Brock W, Mirman L (1972) Optimal economic growth and uncertainty: the discounted case. J Econ Theory 4:479–513
14. Caplin A, Leahy J (2006) The recursive approach to time inconsistency. J Econ Theory 131(1):134–156
15. Carroll K (2005) The method of endogenous grid points for solving dynamic stochastic optimal problems. Econ Lett 91:312–320
16. Chatterjee S, Eyigungor B (2014) Continuous Markov equilibria with quasi-geometric discounting. Working Papers 14–6, Federal Reserve Bank of Philadelphia
17. Cole H, Kubler F (2012) Recursive contracts, lotteries and weakly concave pareto sets. Rev Econ Dyn 15(4):479–500
18. Coleman W (1991) Equilibrium in a production economy with an income tax. Econometrica 59(4):1091–1104
19. Coleman W (2000) Uniqueness of an equilibrium in infinite-horizon economies subject to taxes and externalities. J Econ Theory 95(1):71–78
20. Datta M, Mirman L, Reffett K (2002) Existence and uniqueness of equilibrium in distorted dynamic economies with capital and labor. J Econ Theory 103(2):377–410
21. Fella G (2014) A generalized endogenous grid method for non-smooth and non-concave problems. Rev Econ Dyn 17(2):329–344
22. Feng Z, Miao J, Peralta-Alva A, Santos M (2014) Numerical simulation of nonoptimal dynamic equilibrium models. Int Econ Rev 55:83–110
23. Gong L, Smith W, Zou H (2011) Asset prices and hyperbolic discounting. CEMA Working Papers 486
24. Harris C, Laibson D (2001) Dynamic choices of hyperbolic consumers. Econometrica 69(4):935–959
25. Judd K (2004) Existence, uniqueness, and computational theory for time consistent equilibria: a hyperbolic discounting example (Manuscript)
26. Joshi S (1997) Turnpike theorems in nonconvex nonstationary environments. Int Econ Rev 38:245–248
27. Kocherlakota NR (1996) Reconsideration-proofness: a refinement for infinite horizon time inconsistency. Games Econ Behav 15(1):33–54
28. Kolokoltsov V, Yang W (2012) The turnpike theorems for Markov games. Dyn Games Appl 2(3):294–312
29. Krusell P, Smith A (2000) Consumption-savings decisions with quasi-geometric discounting. CEPR discussion paper no. 2651
30. Krusell P, Smith A (2003) Consumption-savings decisions with quasi-geometric discounting. Econometrica 71:365–375
31. Krusell P, Smith A (2008) Consumption-savings decisions with quasi-geometric discounting: the case with a discrete domain (Manuscript)
32. Krusell P, Kuruşçu B, Smith A (2002) Equilibrium welfare and government policy with quasi-geometric discounting. J Econ Theory 105(1):42–72
33. Kydland F, Prescott E (1980) Dynamic optimal taxation, rational expectations and optimal control. J Econ Dyn Control 2(1):79–91
34. Laibson D (1997) Golden eggs and hyperbolic discounting. Q J Econ 112(2):443–477
35. Laibson D, Repetto A, Tobacman J (1998) Self-control and saving for retirement. Brookings Pap Econ Act 1:91–172
36. Leininger W (1986) The existence of perfect equilibria in a model of growth with altruism between generations. Rev Econ Stud LIII:349–367
37. Luttmer E, Mariotti T (2003) Subjective discounting in an exchange economy. J Polit Econ 111(5):959–989
38. Majumdar M, Zilcha I (1987) Optimal growth in a stochastic environment: some sensitivity and turnpike results. J Econ Theory 43:116–133
39. Maliar L, Maliar S (2003) Solving the neoclassical growth model with quasi-geometric discounting: non-linear Euler-equation methods, IVIE WP AD 2003–23
40. Maliar L, Maliar S (2004) Quasi-geometric discounting: a closed-form solution under the exponential utility function. Bull Econ Res 56(2):201–206
41. Maliar L, Maliar S (2005) Solving the neoclassical growth model with quasi-geometric discounting: a grid-based Euler-equation method. Comput Econ 26:163–172
42. Maliar L, Maliar S (2006a) The neoclassical growth model with heterogeneous quasi-geometric consumers. J Money Credit Bank 38(3):635–654
43. Maliar L, Maliar S (2006b) Indeterminacy in a log-linearized neoclassical growth model with quasi-geometric discounting. Econ Model 23(3):492–505
44. Maliar L, Maliar S (2007) Short-run patience and wealth distribution. Stud Nonlinear Dyn Econ 11(1), Article 4
45. Maliar L, Maliar S (2013) Envelope condition method versus endogenous grid method for solving dynamic programming problems. Econ Lett 120:262–266

46. Maliar L, Maliar S (2014) Numerical methods for large scale dynamic economic models. In: Schmedders K, Judd K (eds) Handbook of computational economics, chapter 7, vol 3. Elsevier Science, Amsterdam
47. Maliar L, Maliar S, Taylor JB, Tsener I (2015) A tractable framework for analyzing a class of nonstationary Markov models. NBER 21155
48. Manuelli R, Sargent T (1987) Exercises in dynamic macroeconomic theory. Harvard University Press, Cambridge
49. Marcet A, Marimon R (1998) Recursive contracts, economics Working Papers, European University Institute
50. Maskin E, Tirole J (2001) Markov perfect equilibrium: I. Observable actions. J Econ Theory 100(2):191–219
51. McKenzie L (1976) Turnpike theory. Econometrica 44:841–865
52. Messner M, Pavoni N, Sleet C (2012) Recursive methods for incentive problems. Rev Econ Dyn 15(4):501–525
53. Mirman L, Morand O, Reffett K (2008) A qualitative approach to markovian equilibrium in infinite horizon economies with capital. J Econ Theory 139(1):75–98
54. Mitra T, Nyarko Y (1991) On the existence of optimal processes in nonstationary environments. J Econ 53:245–270
55. O'Donoghue T, Rabin M (1999) Doing it now or later. Am Econ Rev 89(1):103–124
56. Pavoni N, Sleet C, Messner M (2015) The dual approach to recursive optimization: theory and examples, 2014 meeting papers 1267, society for economic dynamics
57. Peleg B, Yaari M (1973) On the existence of a consistent course of action when tastes are changing. Rev Econ Stud 40(3):391–401
58. Phelps ES, Pollak RA (1968) On second-best national saving and game-equilibrium growth. Rev Econ Stud 35:185–199
59. Pollak RA (1968) Consistent planning. Rev Econ Stud 35:201–208
60. Rustichini A (1998) Dynamic programming solution of incentive constrained problems. J Econ Theory 78(2):329–354
61. Sorger G (2004) Consistent planning under quasi-geometric discounting. J Econ Theory 118(1):118–129
62. Santos M (1999) Numerical solution of dynamic economic models. In: Taylor J, Woodford M (eds) Handbook of macroeconomics. Elsevier Science, Amsterdam, pp 312–382
63. Stachurski J (2009) Economic dynamics: theory and computation. MIT Press, Cambridge
64. Stokey NL, Lucas RE Jr, Prescott E (1989) Recursive methods in economic dynamics. Harvard University Press, Cambridge
65. Strotz RH (1955–1956) Myopia and inconsistency in dynamic utility maximization. Rev Econ Stud 23:165–180
66. Tauchen G (1986) Finite state Markov chain approximations to univariate and vector autoregressions. Econ Lett 20:177–181
67. White M (2015) The method of endogenous gridpoints in theory and practice. Working Papers 15-03, University of Delaware