DISCUSSION PAPER SERIES

DP14025

WHEN THE U.S. CATCHES A COLD, CANADA SNEEZES: A LOWER-BOUND TALE TOLD BY DEEP LEARNING

Vadym Lepetyuk, Lilia Maliar and Serguei Maliar

MONETARY ECONOMICS AND FLUCTUATIONS

ISSN 0265-8003

WHEN THE U.S. CATCHES A COLD, CANADA SNEEZES: A LOWER-BOUND TALE TOLD BY DEEP LEARNING

Vadym Lepetyuk, Lilia Maliar and Serguei Maliar

Discussion Paper DP14025
Published 25 September 2019
Submitted 18 September 2019

Centre for Economic Policy Research
33 Great Sutton Street, London EC1V 0DX, UK
Tel: +44 (0)20 7183 8801
www.cepr.org

WHEN THE U.S. CATCHES A COLD, CANADA SNEEZES: A LOWER-BOUND TALE TOLD BY DEEP LEARNING

Abstract

The Canadian economy was not initially hit by the 2007-2009 Great Recession but ended up having a prolonged episode of the effective lower bound (ELB) on nominal interest rates. To investigate the Canadian ELB experience, we build a "baby" ToTEM model -- a scaled-down version of the Terms of Trade Economic Model (ToTEM) of the Bank of Canada. Our model includes 49 nonlinear equations and 21 state variables. To solve such a high-dimensional model, we develop a projection deep learning algorithm -- a combination of unsupervised and supervised (deep) machine learning techniques. Our findings are as follows: The Canadian ELB episode was contaminated from abroad via large foreign demand shocks. Prolonged ELB episodes are easy to generate in open-economy models, unlike in closed-economy models. Nonlinearities associated with the ELB constraint have virtually no impact on the Canadian economy but other nonlinearities do, in particular, the degree of uncertainty and specific closing condition used to induce the model's stationarity.

JEL Classification: C61, C63, C68, E31, E52

Keywords: central banking, ToTEM, Machine Learning, deep learning, supervised learning, neural networks, clustering analysis large-scale model, New Keynesian Model, ZLB

Vadym Lepetyuk - lepetyuk@gmail.com
Bank of Canada

Lilia Maliar - lmaliar@gc.cuny.edu
The Graduate Center, City University of New York and CEPR

Serguei Maliar - smaliar@scu.edu
Santa Clara University and Columbia University

Acknowledgements
The authors acknowledge the support from the Bank of Canada and useful comments from Stefano Gnocchi, the senior research director in the Canadian economic analysis department. The authors are especially grateful for help, encouragement and constructive comments of the former head of Bank of Canada's model development division Oleksiy Kryvtsov, who initiated this project, and the current head of the division José Dorich, who led this project and brought it to its completion. The authors are also thankful to HanJing Xie for excellent assistance.

When the U.S. catches a cold, Canada sneezes: a lower-bound tale told by deep learning*

Vadym Lepetyuk, Lilia Maliar and Serguei Maliar

September 25, 2019

Abstract

The Canadian economy was not initially hit by the 2007-2009 Great Recession but ended up having a prolonged episode of the effective lower bound (ELB) on nominal interest rates. To investigate the Canadian ELB experience, we build a "baby" ToTEM model – a scaled-down version of the Terms of Trade Economic Model (ToTEM) of the Bank of Canada. Our model includes 49 nonlinear equations and 21 state variables. To solve such a high-dimensional model, we develop a projection deep learning algorithm – a combination of unsupervised and supervised (deep) machine learning techniques. Our findings are as follows: The Canadian ELB episode was contaminated from abroad via large foreign demand shocks. Prolonged ELB episodes are easy to generate in open-economy models, unlike in closed-economy models. Nonlinearities associated with the ELB constraint have virtually no impact on the Canadian economy but other nonlinearities do, in particular, the degree of uncertainty and specific closing condition used to induce the model's stationarity.

JEL classification: C61, C63, C68, E31, E52

Keywords: central banking, policymakers, ToTEM; bToTEM; stochastic simulation; machine learning; deep learning; supervised learning; unsupervised learning; neural networks; ergodic set; clustering analysis; adaptive grid; central bank; Bank of Canada, US Fed; large-scale model; new Keynesian model; ZLB; ELB

*Corresponding author: Lilia Maliar, The Graduate Center, CUNY, NY 10016, USA. Email: lmaliar@gc.cuny.edu. This paper is a revised version of Bank of Canada staff paper #2017-21. The authors acknowledge the support from the Bank of Canada and useful comments from Stefano Gnocchi, the senior research director in the Canadian economic analysis department. The authors are especially grateful for help, encouragement and constructive comments of the former head of Bank of Canada's model development division Oleksiy Kryvtsov, who initiated this project, and the current head of the division José Dorich, who led this project and brought it to its completion. The authors are also thankful to HanJing Xie for excellent assistance. The views expressed in this paper are solely those of the authors and may differ from official Bank of Canada views. No responsibility for them should be attributed to the Bank of Canada.

1 Introduction

The Canadian economy did not experience a 2007 subprime crisis and was not initially hit by the Great Recession, unlike the U.S. and Europe; see a speech of the governor Boivin (2011). Nonetheless, after few months, Canada entered a prolonged episode of the effective lower bound (ELB) on nominal interest rates.[1] In the paper, we investigate a hypothesis that the ELB crisis was contaminated to Canada from abroad, in particular, from the US.

Bank of Canada has a well-developed macroeconomic model of the Canadian economy called the Terms of Trade Economic Model (ToTEM); see a technical report of Dorich et al. (2013). The model is huge – 356 equations and unknowns and 215 state variables – and is analyzed exclusively by first-order perturbation methods. In the paper, we construct a scaled-down version of ToTEM, which we call a "baby" ToTEM (bToTEM). Our model is still very large: it includes 49 equations and 21 state variables. We introduce a deep learning (DL) projection algorithm – a combination of unsupervised and supervised machine learning techniques – capable of constructing fully nonlinear solutions.

We calibrate the bToTEM model by following the ToTEM analysis as closely as possible. We find that our scaled-down model replicates remarkably well the impulse response functions of the full-scale ToTEM model. We then conduct two empirically relevant policy experiments related to a recent episode of ELB on nominal interest rates during the Great Recession.

In the first experiment, we introduce into bToTEM a plausible sequence of ROW variables estimated from actual data using ToTEM. In the second experiment, we analyze a change in the inflation target from 2 to 3 percent by holding the real interest rate fixed according to the practice of the Bank of Canada. We compare our DL nonlinear solutions to first- and second-order perturbation solutions; we obtained these solutions from Dynare and from an IRIS toolbox; the latter is a first-order toolbox capable of dealing with occasionally binding inequality constraints.

Our analysis delivers several interesting findings: First, we show that international transmission of ELB is an empirically plausible mechanism for explaining the Canadian ELB experience. Specifically, in the beginning of the Great Recession, Canada faced a dramatic reduction in foreign demand (in particular, in the U.S. demand), and it proved sufficient to produce a prolonged ELB episode in the bToTEM model. Thus, our analysis contributes to the literature on the international transmission of a liquidity trap by offering a new mechanism of the ELB contamination. A distinctive feature of our analysis is that it is carried out in a realistic and meticulously calibrated model of the Canadian economy under plausible foreign demand shocks.[2]

Second, our analysis reveals that it is surprisingly easy to generate prolonged ELB (or ZLB) episodes in an open-economy setting via international shocks, while it is virtually impossible to produce such episodes in closed-economy models under meaningful calibrations; see Fernández-Villaverde et al. (2012, 2015) and Christiano et al. (2015) for a discussion. For example, in order to obtain realistic spells at the ZLB, Aruoba et al. (2018) augment the simulated series from the model to include historical data from the U.S. economy because the model cannot generate such data otherwise.

Third, we find that the Canadian economy would entirely avoid the ELB episode if the target inflation rate were 3 instead of 2 percent. This finding is important because the possibility of a new ELB episode remains an important practical issue for the Bank of Canada; see Dorich et al. (2018).

Fourth, we find that the ELB restriction plays virtually no role in bToTEM's performance. In particular, both local and global solution methods predict very similar timing and duration of the ELB episodes. In simple words, the presence of active ZLB does not visibly affect the model's variables other than the interest

[1] ELB is similar to ZLB on (net) nominal interest rates but it is set at a level other than zero. What is important in both cases is that there is a lower bound that becomes binding.

[2] The literature documenting the importance of foreign shocks for the business cycle propagation includes Backus et al. (1992), Schmitt-Grohé (1998), Lubik and Schorfheide (2005); see also Fernández et al. (2017) for recent evidence. The literature analyzing the transmission of liquidity trap from one country to another includes Fujiwara (2010), Jeanne (2010), Bodenstein et al.(2016), Cook and Devereux (2011, 2013, 2016), Corsetti et al. (2016), Devereux (2014), Caballero et al. (2016), and Eggertsson et al. (2016), among others.

rate. Our findings support the ZLB irrelevance hypothesis documented in the recent empirical literature; see Debortoli et al. (2019) for a review.

Fifth, we discover that other nonlinearities – those not associated with ELB – play an important role in the model's predictions. In particular, when assessing the impact of a hypothetical transition from a 2 to 3 percent inflation target on the Canadian economy, we spot economically significant differences between linear and nonlinear solutions. We show that such differences are attributed mostly to the uncertainty effect which leads to a large difference in steady states between linearized and nonlinear solutions. We show how to control for the uncertainty effect: if each solution starts the transition from its own steady state (which we view as a coherent approach), the linear and nonlinear solutions look like vertical shifts of one another.

Finally and most strikingly, we find that the closing condition, which is used to induce stationarity in open-economy models, plays a critical role in the bToTEM dynamics. To close the model, we assume that the risk premium on foreign bonds decreases with the quantity of bonds purchased, which is one of the approaches introduced in Schmitt-Grohé and Uribe (2003). When we used a closing condition in the linear form, the predictions of linearized and nonlinear models were similar. However, when we switch to a similar condition in an exponential form, as in Schmitt-Grohé and Uribe (2003), the predictions of the linearized and nonlinear models were dramatically different. To the best of our knowledge, we are the first to document such a difference. Schmitt-Grohé and Uribe (2003) and the following papers find that a particular closing condition does not practically affect quantitative implications of open-economy models.

To gain intuition into this finding, we revisit the analysis of Schmitt-Grohé and Uribe (2003). Our initial guess was that the closing condition also matters in their model but their linearization analysis simply was not able to detect it (because the linearized versions of the linear and exponential closing conditions are identical). To check on this guess, we construct high-order perturbation and global DL solutions for the model of Schmitt-Grohé and Uribe (2003) as well. Our numerical findings did not support our initial guess. Specifically, the predictions of the model of Schmitt-Grohé and Uribe (2003) were very similar under the linear and exponentiated closing conditions, except for the foreign debt accumulation, which was dampened under some parameterizations – this was true for both perturbation and nonlinear DL solutions. That is, in the model of Schmitt-Grohé and Uribe (2003), we do not observe the quantitative importance of high-order terms, which we encounter in bToTEM.

DL was critical for telling the tale of the Canadian ELB episode. The bToTEM model is intractable under conventional value function iteration or nonlinear projection methods because of the curse of dimensionality. In the earlier version of the paper, namely, Lepetyuk et al. (2017), we used unsupervised machine learning (clustering) for constructing the solution domain but we approximated the decision functions with polynomial functions, identical to those used by a second-order perturbation method. We found that polynomial functions are not flexible enough to accurately approximate highly nonlinear solutions, and we opted to scale down the volatility of shocks to achieve the convergence. As a result, the effects of nonlinearity were modest. In the present paper, we introduce a projection DL algorithm in which unsupervised learning is complemented with supervised learning, namely, we parameterize decision functions with neural networks which we train by using a stochastic gradient descent method.[3] The introduction of DL both increases the accuracy and enhances the convergence which enable us to solve the bToTEM model under the empirically relevant volatility.[4] This is precisely what magnifies the effects of nonlinearity on the solution in the present paper.

The rest of the paper is organized as follows: In Section 2, we construct the bToTEM model. In Section 3, we outline the calibration procedure of the bToTEM model and compare its impulse response

[3] In the present paper, we use neural networks for approximating decision functions. Maliar et al. (2018, 2019) introduce a different DL method in which the entire economic model is reformulated as an objective function for training (in the form of lifetime reward and residuals in the Bellman and Euler equations). The machine is then trained to optimize the objective function with the tools from the Google's TensorFlow library.

[4] While working on the paper, we learned that several other recent papers successfully used neural networks for constructing their numerical solutions including Duarte (2018), Villa and Valaitis (2019), Fernández-Villaverde et al. (2019), Azinović et al. (2019).

functions with the ToTEM and LENS models. In Section 4, we describe the implementation of nonlinear solution methods for the bToTEM model, and we compare the properties of linear and nonlinear solutions. In Section 5, we use the bToTEM model to analyze the response of the Canadian economy to foreign demand shocks. In Section 6, we simulate an increase of the inflation target from 2 to 3 percent. In Section 7, we analyze the role of the closing condition. Finally, in Section 8, we conclude.

2 The bToTEM model

Nowadays, central banks, as well as leading international organizations and government agencies, use large-scale macroeconomic models in practical policymaking. Notable examples are the International Monetary Fund's Global Economy Model, GEM (Bayoumi et al., 2001), the U.S. Federal Reserve Board's SIGMA model (Erceg et al., 2006), the Bank's of Canada Terms of Trade Economic Model, ToTEM (Dorich et al., 2013), the European Central Bank's New Area-Wide Model, NAWM (Coenen et al., 2008), the Bank of England's COMPASS Model (Burgess et al., 2013) and the Swedish Riksbank's Ramses II Model (Adolfson et al., 2013). These are general equilibrium models that typically include several types of utility-maximizing consumers, several profit-maximizing production sectors, fiscal and monetary authorities, as well as a foreign sector.[5] Central banks' models must be rich and flexible enough to realistically describe the interactions between numerous variables that are of interest to policymakers, including different types of foreign and domestic inputs, outputs, consumption, investment, capital, labor, prices, exchange rate, as well as monetary variables and financial assets. Some central banking models contain hundreds of equations and unknowns! The goal of these models is to mimic as closely as possible the actual economies in every possible dimension of interest. Using such rich models, policymakers can produce realistic macroeconomic projections and promptly analyze the consequences of alternative policies.

2.1 bToTEM versus ToTEM

ToTEM is the main projection and policy analysis model of the Bank of Canada.[6] This is a large-scale general equilibrium macroeconomic model that currently contains 356 equations and unknowns. We construct and calibrate bToTEM – a scaled down version of ToTEM that has 49 equations and unknowns, including 21 state variables. In the construction of bToTEM, we follow ToTEM as closely as possible. Like the full-scale ToTEM model, the bToTEM is a small open-economy model that features the new-Keynesian Phillips curves for consumption, labor and imports. As in ToTEM, we assume the rule-of-thumb price settlers in line with Galí and Gertler (1999). We use a quadratic adjustment cost of investment and a convex cost of capital utilization. We maintain the ToTEM's terms of trade assumption; namely, we allow for bidirectional trade consisting of exporting domestic consumption goods and commodities, and importing foreign goods for domestic production.

There are three aspects in which bToTEM is simplified relatively to ToTEM. First, the full-scale ToTEM model consists of five distinct production sectors, namely, those for producing consumption goods and services, investment goods, government goods, noncommodity export goods, commodities, and it also has a separate economic model of the rest of the world (ROW). The first four of ToTEM's production sectors have identical production technology and constraints, and only differ in the values of parameters. In the bToTEM model, in place of the four sectors we assume just one production sector, which is identical in structure to the consumption goods and services sector of the ToTEM model, and we introduce linear technologies for transforming the output of this sector into other types of output corresponding to the remaining ToTEM's sectors.

[5] There are also less structural large-scale microeconometric models of central banking, e.g., FRB of the U.S. Federal Reserve (Brayton, 1997) and LENS of the Bank of Canada (Gervais and Gosselin, 2014).

[6] See Dorich et al. (2013) for a detailed technical report on the ToTEM model. This model – known also as TOTEM II – builds on the original ToTEM model in Murchison and Renisson (2006); see also Binette et al. (2004) for an earlier simplified version of the original ToTEM model.

Second, there are three types of households in ToTEM that differ in their saving opportunities. In turn, in bToTEM we assume just one type of household. Like in ToTEM, the bToTEM's households supply differentiated labor services in exchange for sticky wages. Under our assumptions, Phillips curves in bToTEM are identical to those in ToTEM; the difference is that bToTEM has three Phillips curves, while ToTEM has eight Phillips curves.

Finally, in ToTEM, the ROW sector is represented as a separate new Keynesian model with its own production sector, while in bToTEM the ROW sector is modeled by using appropriately calibrated exogenous processes for foreign variables.

In the main text, we describe the optimization problems of economic agents and the key model's equations in bToTEM; the derivation of equilibrium conditions and a list of the model's equations are provided in Appendices A and C, respectively.

2.2 Production of final goods

The production sector of the economy consists of two stages. In the first stage, intermediate goods are produced by identical perfectly competitive firms from labor, capital, commodities, and imports. In the second stage, a variety of final goods are produced by monopolistically competitive firms from the intermediate goods. The final goods are then aggregated into the final consumption good.

First stage of production. In the first stage of production, the representative, perfectly competitive firm produces an intermediate good using the following constant elasticity of substitution (CES) technology:

$$Z_t^g = \left[\delta_l \left(A_t L_t\right)^{\frac{\sigma-1}{\sigma}} + \delta_k \left(u_t K_t\right)^{\frac{\sigma-1}{\sigma}} + \delta_{com} \left(COM_t^d\right)^{\frac{\sigma-1}{\sigma}} + \delta_m \left(M_t\right)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}, \tag{1}$$

where L_t, K_t, and COM_t^d are labor, capital and commodity inputs, respectively, M_t is imports, u_t is capital utilization, and A_t is the level of labor-augmenting technology that follows a stochastic process given by

$$\log A_t = \varphi_a \log A_{t-1} + (1-\varphi_a) \log \bar{A} + \xi_t^a, \tag{2}$$

with ξ_t^a being a normally distributed variable, and φ_a being an autocorrelation coefficient.

Capital depreciates according to the following law of motion:

$$K_{t+1} = (1-d_t) K_t + I_t, \tag{3}$$

where d_t is the depreciation rate, and I_t is investment. The depreciation rate increases with capital utilization as follows:

$$d_t = d_0 + \overline{d} e^{\rho(u_t - 1)}. \tag{4}$$

The firm incurs a quadratic adjustment cost when adjusting the level of investment. The net output is given by

$$Z_t^n = Z_t^g - \frac{\chi_i}{2} \left(\frac{I_t}{I_{t-1}} - 1\right)^2 I_t. \tag{5}$$

The objective of the firm is to choose L_t, K_{t+1}, I_t, COM_t, M_t, u_t in order to maximize profits

$$E_0 \sum_{t=0}^{\infty} \mathcal{R}_{0,t} \left(P_t^z Z_t^n - W_t L_t - P_t^i I_t - P_t^{com} COM_t^d - P_t^m M_t\right)$$

subject to (1)–(5). The firm discounts nominal payoffs according to household's stochastic discount factor $\mathcal{R}_{t,t+j} = \beta^j \left(\lambda_{t+j}/\lambda_t\right) \left(P_t/P_{t+j}\right)$, where λ_t is household's marginal utility of consumption and P_t is the final good price.

Second stage of production. In the second stage of production, a continuum of monopolistically competitive firms indexed by i produce differentiated goods from the intermediate goods and manufactured inputs. The production technology features perfect complementarity

$$Z_{it} = \min\left(\frac{Z_{it}^{n}}{1-s_m}, \frac{Z_{it}^{mi}}{s_m}\right),$$

where Z_{it}^{n} is an intermediate good and Z_{it}^{mi} is a manufactured input, and s_m is a Leontief parameter. The differentiated goods Z_{it} are aggregated into the final good Z_t according to the following CES technology:

$$Z_t = \left(\int_0^1 Z_{it}^{\frac{\varepsilon-1}{\varepsilon}} di\right)^{\frac{\varepsilon}{\varepsilon-1}}.$$

Cost minimization implies the following demand function for a differentiated good i:

$$Z_{it} = \left(\frac{P_{it}}{P_t}\right)^{-\varepsilon} Z_t, \tag{6}$$

where

$$P_t = \left(\int_0^1 P_{it}^{1-\varepsilon} di\right)^{\frac{1}{1-\varepsilon}}. \tag{7}$$

The final good is used as the manufactured inputs by each of the monopolistically competitive firms.

There are monopolistically competitive firms of two types: rule-of-thumb firms of measure ω and forward-looking firms of measure $1-\omega$. Within each type with probability θ the firms index their price to the inflation target $\bar{\pi}_t$ as follows: $P_{it} = \bar{\pi}_t P_{i,t-1}$. With probability $1-\theta$, the rule-of-thumb firms partially index their price to lagged inflation and target inflation according to the following rule:

$$P_{it} = (\pi_{t-1})^{\gamma} (\bar{\pi}_t)^{1-\gamma} P_{i,t-1}. \tag{8}$$

The forward-looking firms with probability $1-\theta$ choose their price P_t^* in order to maximize profits generated when the price remains effective

$$\max_{P_t^*} E_t \sum_{j=0}^{\infty} \theta^j \mathcal{R}_{t,t+j} \left(\prod_{k=1}^{j} \bar{\pi}_{t+k} P_t^* Z_{i,t+j} - (1-s_m) P_{t+j}^{z} Z_{i,t+j} - s_m P_{t+j} Z_{i,t+j}\right) \tag{9}$$

subject to demand constraints

$$Z_{i,t+j} = \left(\frac{\prod_{k=1}^{j} \bar{\pi}_{t+k} P_t^*}{P_{t+j}}\right)^{-\varepsilon} Z_{t+j}. \tag{10}$$

Relation between the first and second stages of production. The production in the first and second stages are related as follows:

$$Z_t^n = \int_0^1 Z_{it}^n di = (1-s_m) \int_0^1 Z_{it} di = (1-s_m) \int_0^1 \left(\frac{P_{it}}{P_t}\right)^{-\varepsilon} Z_t di = (1-s_m) \Delta_t Z_t, \tag{11}$$

where $\Delta_t = \int_0^1 \left(\frac{P_{it}}{P_t}\right)^{-\varepsilon} di$ is known as *price dispersion.*

Finally, in order to maintain the relative prices of the investment goods and noncommodity exports in accordance to the national accounts, these goods are assumed to be produced from the final goods according to linear technology that implies $P_t^i = \iota_i P_t$ and $P_t^{nc} = \iota_x P_t$, where P_t^i and P_t^{nc} are the price of investment goods and noncommodity exports goods, respectively.

2.3 Commodities

The representative, perfectly competitive domestic firm produces commodities using final goods according to the following CES technology:

$$COM_t = (Z_t^{com})^{s_z} (A_t F)^{1-s_z} - \frac{\chi_{com}}{2} \left(\frac{Z_t^{com}}{Z_{t-1}^{com}} - 1 \right)^2 Z_t^{com}, \tag{12}$$

where Z_t^{com} is the final good input, and F is a fixed production factor, which may be considered as land. Similarly to production of final goods, the commodity producers incur quadratic adjustment costs when they adjust the level of final good input.

The commodities are sold domestically (COM_t^d) or exported to the rest of the world (X_t^{com})

$$COM_t = COM_t^d + X_t^{com}.$$

They are sold at the world price adjusted by the nominal exchange rate as follows:

$$P_t^{com} = e_t P_t^{comf},$$

where e_t is the nominal exchange rate (i.e., domestic price of a unit of foreign currency), and P_t^{comf} is the world commodity price. In real terms, the latter price is given by

$$p_t^{com} = s_t p_t^{comf}, \tag{13}$$

where $p_t^{com} \equiv P_t^{com}/P_t$ and $p_t^{comf} \equiv P_t^{comf}/P_t^f$ are domestic and foreign relative prices of commodities, respectively, P_t^f is the foreign consumption price level, and $s_t = e_t P_t^f / P_t$ is the real exchange rate.

2.4 Imports

The final imported good M_t is bonded from intermediate imported goods according to the following technology:

$$M_t = \left(\int_0^1 M_{it}^{\frac{\varepsilon_m - 1}{\varepsilon_m}} di \right)^{\frac{\varepsilon_m}{\varepsilon_m - 1}},$$

where M_{it} is an intermediate imported good i. The demand for an intermediate imported good i is given by

$$M_{it} = \left(\frac{P_{it}^m}{P_t^m} \right)^{-\varepsilon_m} M_t,$$

where

$$P_t^m = \left(\int_0^1 (P_{it}^m)^{1-\varepsilon_m} di \right)^{\frac{1}{1-\varepsilon_m}}.$$

We assume the prices of the intermediate imported goods to be sticky in a similar way as the prices of the differentiated final goods. A measure ω_m of the importers follows the rule-of-thumb pricing, and the others are forward looking. The optimizing forward-looking importers choose the price P_t^{m*} in order to maximize profits generated when the price remains effective

$$\max_{P_t^{m*}} E_t \sum_{j=0}^{\infty} (\theta_m)^j \mathcal{R}_{t,t+j} \left(\prod_{k=1}^{j} \bar{\pi}_{t+k} P_t^{m*} M_{i,t+j} - e_{t+j} P_{t+j}^{mf} M_{i,t+j} \right)$$

subject to demand constraints

$$M_{i,t+j} = \left(\frac{\prod_{k=1}^{j} \bar{\pi}_{t+k} P_t^{m*}}{P_{t+j}^m} \right)^{-\varepsilon_m} M_{t+j},$$

where P_t^{mf} is the price of imports in the foreign currency. All importers face the same marginal cost given by the foreign price of imports.

2.5 Households

The representative household in the economy has the period utility function over consumption of finished goods and a variety of differentiated labor service

$$U_t = \frac{\mu}{\mu-1}\left(C_t - \xi \bar{C}_{t-1}\right)^{\frac{\mu-1}{\mu}} \exp\left(\frac{\eta(1-\mu)}{\mu(1+\eta)} \int_0^1 (L_{ht})^{\frac{\eta+1}{\eta}} dh\right) \eta_t^c, \tag{14}$$

where C_t is the household consumption of finished goods; $\bar{C}_t$ is the aggregate consumption, which the representative household takes as given; L_{ht} is labor service of type h; η_t^c is a consumption demand shock that follows a process

$$\log \eta_t^c = \varphi_c \log \eta_{t-1}^c + \xi_t^c, \tag{15}$$

with ξ_t^c being a normally distributed variable, and φ_c being an autocorrelation coefficient.

The representative household of type h maximizes the lifetime utility

$$E_0 \sum_{t=0}^{\infty} \beta^t U_t \tag{16}$$

subject to the following budget constraints:

$$P_t C_t + \frac{B_t}{R_t} + \frac{e_t B_t^f}{R_t^f\left(1+\kappa_t^f\right)} = B_{t-1} + e_t B_{t-1}^f + \int_0^1 W_{ht} L_{ht} dh + \Pi_t, \tag{17}$$

where B_t and B_t^f are holdings of domestic and foreign-currency denominated bonds, respectively; R_t and R_t^f are domestic and foreign nominal interest rate, respectively; κ_t^f is the risk premium on the foreign interest rate; W_{ht} is the nominal wage of labor of type h; Π_t is profits paid by the firms.

2.6 Wage setting

The representative household supplies a variety of differentiated labor service to the labor market, which is monopolistically competitive. The differentiated labor service is aggregated according to the following aggregation function:

$$L_t = \left(\int_0^1 L_{ht}^{\frac{\varepsilon_w - 1}{\varepsilon_w}} dh\right)^{\frac{\varepsilon_w}{\varepsilon_w - 1}}.$$

Aggregated labor L_t is demanded by firms in the first stage of production. A cost minimization of the aggregating firm implies the following demand for individual labor:

$$L_{ht} = \left(\frac{W_{ht}}{W_t}\right)^{-\varepsilon_w} L_t, \tag{18}$$

where W_{ht} is wage for labor service of type h, and W_t is defined by the following:

$$W_t \equiv \left(\int_0^1 W_{ht}^{1-\varepsilon_w} dh\right)^{\frac{1}{1-\varepsilon_w}}. \tag{19}$$

Wages are set by labor unions that are of two types: rule-of-thumb unions of measure ω_w and forward-looking unions of measure $1-\omega_w$. Within each type, with probability θ_w the labor unions index their wage to the inflation target $\bar{\pi}_t$ as follows $W_{it} = \bar{\pi} W_{i,t-1}$. The rule-of-thumb unions that do not index their wage in the current period follow the rule

$$W_{it} = \left(\pi_{t-1}^w\right)^{\gamma_w} (\bar{\pi}_t)^{1-\gamma_w} W_{i,t-1}. \tag{20}$$

The forward-looking unions that do not index their wage choose the wage W_t^* optimally in order to maximize the household utility function when the wage is effective

$$E_t \sum_{j=0}^{\infty} (\beta\theta_w)^j \, U_{t+j} \tag{21}$$

subject to labor demand (18) written as

$$L_{h,t+j} = \left(\frac{\prod_{k=1}^{j} \bar{\pi}_{t+k} W_t^*}{W_{t+j}} \right)^{-\varepsilon_w} L_{t+j}, \tag{22}$$

and budget constraints (17) which can be written as

$$P_{t+j}C_{t+j} = \prod_{k=1}^{j} \bar{\pi}_{t+k} W_t^* L_{h,t+j} dh + \Psi_{t+j},$$

where Ψ_{t+j} includes terms other than C_{t+j} and $L_{h,t+j}$.

2.7 Monetary policy

The monetary authority sets the short-term nominal interest rate in response to a deviation of the actual inflation rate from the target and a deviation of the actual output from potential output,

$$\Phi_t = \rho_r R_{t-1} + (1 - \rho_r) \left[\bar{R} + \rho_\pi (\pi_t - \bar{\pi}_t) + \rho_Y \left(\log Y_t - \log \bar{Y}_t \right) \right] + \eta_t^r, \tag{23}$$

where ρ_r measures the degree of smoothing of the interest rate; $\bar{R}$ is the long-run nominal interest rate; ρ_π measures a long-run response to the inflation gap; $\bar{\pi}_t$ is the inflation target; ρ_Y measures a long-run response to the output gap; $\bar{Y}_t$ is the potential level of output; η_t^r is an interest rate shock that is assumed to follow the following process:

$$\eta_t^r = \varphi_r \eta_{t-1}^r + \xi_t^r,$$

where ξ_t^r is a normally distributed variable, and φ_r is an autocorrelation coefficient. Potential output changes with productivity in the following stylized way:

$$\log \bar{Y}_t = \varphi_z \log \bar{Y}_{t-1} + (1 - \varphi_z) \log \left(\frac{A_t \bar{Y}}{\bar{A}} \right).$$

If an effective lower bound R_t^{elb} is imposed on the nominal interest rate, the interest rate is determined as a maximum of (23) and R_t^{elb}:

$$R_t = \max \left\{ R_t^{elb}, \Phi_t \right\}.$$

2.8 Foreign demand for noncommodity exports

We assume that the foreign demand for noncommodity exports is given by the following demand function:

$$X_t^{nc} = \gamma^f \left(\frac{P_t^{nc}}{e_t P_t^f} \right)^{-\phi} Z_t^f, \tag{24}$$

where P_t^{nc} is a domestic price of noncommodity exports. In real terms, we have

$$X_t^{nc} = \gamma^f \left(\frac{s_t}{p_t^{nc}} \right)^{\phi} Z_t^f. \tag{25}$$

2.9 Balance of payments

The balance of payments is

$$\frac{e_t B_t^f}{R_t^f\left(1+\kappa_t^f\right)} - e_t B_{t-1}^f = P_t^{nc} X_t^{nc} + P_t^{com} X_t^{com} - P_t^m M_t, \tag{26}$$

where B_t^f is domestic holdings of foreign-currency denominated bonds, and R_t^f is the nominal interest rate on the bonds. In real terms, it becomes

$$\frac{b_t^f}{r_t^f\left(1+\kappa_t^f\right)} - b_{t-1}^f \frac{s_t}{s_{t-1}} = \frac{1}{\bar{Y}}\left(p_t^{nc} X_t^{nc} + p_t^{com} X_t^{com} - p_t^m M_t\right), \tag{27}$$

where the bond holdings are normalized as $b_t^f = \frac{e_t B_t^f}{\pi_{t+1}^f P_t \bar{Y}}$, and r_t^f is the real interest rate on the foreign-currency denominated bonds.

2.10 Rest-of-the-world economy

The rest of the world is specified by three exogenous processes that describe the evolution of foreign variables. First, the foreign output Z_t^f is given by

$$\log Z_t^f = \varphi_{zf} \log Z_{t-1}^f + \left(1-\varphi_{zf}\right) \log \bar{Z}^f + \xi_t^{zf}; \tag{28}$$

second, the foreign real interest rate r_t^f follows

$$\log r_t^f = \varphi_{rf} \log r_{t-1}^f + \left(1-\varphi_{rf}\right) \log \bar{r} + \xi_t^{rf}; \tag{29}$$

finally, a foreign commodity price p_t^{comf} is

$$\log p_t^{comf} = \varphi_{comf} \log p_{t-1}^{comf} + \left(1-\varphi_{comf}\right) \log \bar{p}^{comf} + \xi_t^{comf}, \tag{30}$$

where ξ_t^{zf}, ξ_t^{rf} and ξ_t^{comf} are normally distributed random variables, and φ_{Zf}, φ_{rf} and φ_{comf} are the autocorrelation coefficients.

2.11 Uncovered interest rate parity

We impose an augmented uncovered interest rate parity condition

$$e_t = E_t\left[\left(e_{t-1}\right)^{\varkappa}\left(e_{t+1}\frac{R_t^f\left(1+\kappa_t^f\right)}{R_t}\right)^{1-\varkappa}\right], \tag{31}$$

where the term $(e_{t-1})^{\varkappa}$ under the brackets is added to mimic the relationship assumed in ToTEM; see Appendix A.4 for some more details. Without the augmented term the difference in interest rates between two countries would be equal to the expected change in exchange rates between the countries' currencies.

2.12 Market clearing conditions

We close the model by the following resource feasibility condition:

$$Z_t = C_t + \iota_i I_t + \iota_x X_t^{nc} + Z_t^{com} + \upsilon_z Z_t. \tag{32}$$

We define GDP and GDP deflator as follows:

$$Y_t = C_t + I_t + X_t^{nc} + X_t^{com} - M_t + \upsilon_y Y_t, \tag{33}$$
$$P_t^y Y_t = P_t C_t + P_t^i I_t + P_t^{nc} X_t^{nc} + P_t^{com} X_t^{com} - P_t^m M_t + \upsilon_y P_t^y Y_t;$$

in real terms, the latter becomes

$$p_t^y Y_t = C_t + p_t^i I_t + p_t^{nc} X_t^{nc} + p_t^{com} X_t^{com} - p_t^m M_t + \upsilon_y p_t^y Y_t. \tag{34}$$

2.13 Stationarity condition for the open-economy model

The budget constraint (17) of the domestic economy contains $R_t^f \left(1 + \kappa_t^f\right)$ where R_t^f is the rate of return to foreign assets and κ_t^f is the risk premium. If the rate of return to foreign assets does not depend on quantity purchased, then the domestic economy can maintain nonvanishing long-run growth by investing in foreign assets. Schmitt-Grohé and Uribe (2003) explore several alternative assumptions that make it possible to prevent this undesirable implication and to attain stationarity in open-economy models. We adopt one of their assumptions, namely, we assume that the risk premium κ_t^f is a decreasing function of foreign assets

$$\kappa_t^f = \varsigma \left(\bar{b}^f - b_t^f\right), \tag{35}$$

where $\bar{b}^f$ is the steady state level of the normalized bond holdings. This assumption ensures a decreasing rate of return to foreign assets. As we will see, a specific functional form assumed for modeling risk premium plays an important role in the model's predictions.

3 A comparison of bToTEM to ToTEM and LENS

We describe the calibration procedure for the bToTEM model, and we compare impulse response functions produced by the bToTEM model to those produced by ToTEM and LENS, the two models of the Bank of Canada.

3.1 Calibration of bToTEM

The bToTEM model contains 61 parameters that need to be calibrated. Whenever possible, we use the same values of parameters in bToTEM as those in ToTEM, and we choose the remaining parameters to reproduce a selected set of observations from the Canadian time series data. In particular, our calibration procedure targets the ratios of six nominal variables to nominal GDP $P_t^y Y_t$, namely, consumption $P_t C_t$, investment $P_t^i I_t$, noncommodity export $P_t^{nc} X_t^{nc}$, commodity export $P_t^{com} X_t^{com}$, import $P_t^m M_t$, total commodities $P_t^{com} COM_t$, and labor input $W_t L_t$. Furthermore, we calibrate the persistence of shocks so that the standard deviations of the selected bToTEM variables coincide with those of the corresponding ToTEM variables, namely, those of domestic nominal interest rate R_t, productivity A_t, foreign demand Z_t^f, foreign commodity price p_t^{comf}, and foreign interest rate r_t^f. The parameters choice is summarized in Tables D1 and D2 provided in Appendix D.

3.2 Impulse response functions of bToTEM, ToTEM and LENS

The ToTEM model is analyzed by the Bank of Canada with the help of a first-order perturbation method that is implemented by using IRIS software.[7] To compare our bToTEM with ToTEM, we construct a similar first-order perturbation solution to bToTEM by using both IRIS and Dynare software.[8] We verified that the IRIS and Dynare packages produce indistinguishable numerical solutions to bToTEM).[9]

[7] This software is available at http://www.iris-toolbox.com; see Beneš, Johnston and Plotnikov (2015) for its description.

[8] The IRIS toolbox is also used by the Bank of Canada; see Beneš et al. (2015) for IRIS documentation; and see also Laséen and Svensson (2011), Guerrieri and Iacoviello (2015) and Holden (2016) for related methods.

[9] Dynare software is available at http://www.dynare.org; see Adjemian et al. (2011) for the documentation.

In the comparison analysis, we also include the impulse response functions for the LENS model, which is another model of the Bank of Canada.

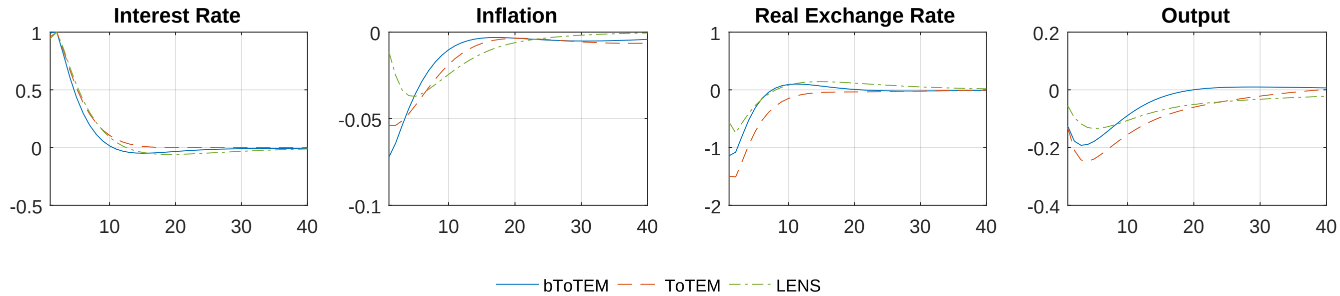

Figure 1: Impulse response functions: interest rate shock

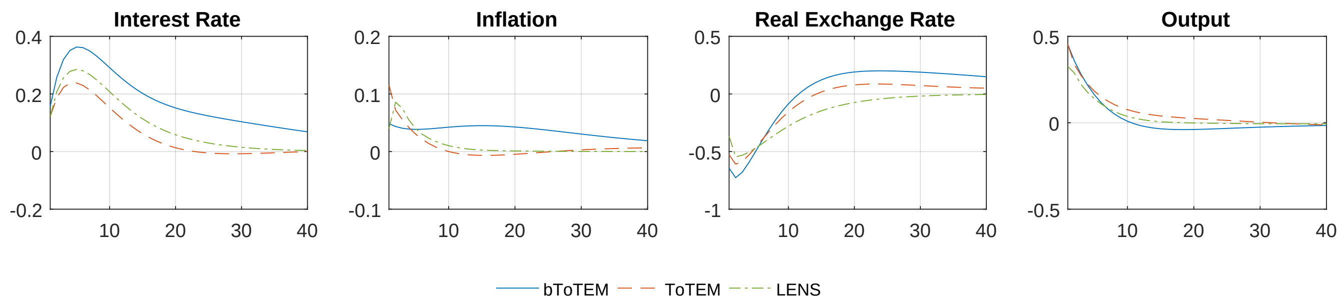

Figure 2: Impulse response functions: consumption demand shock

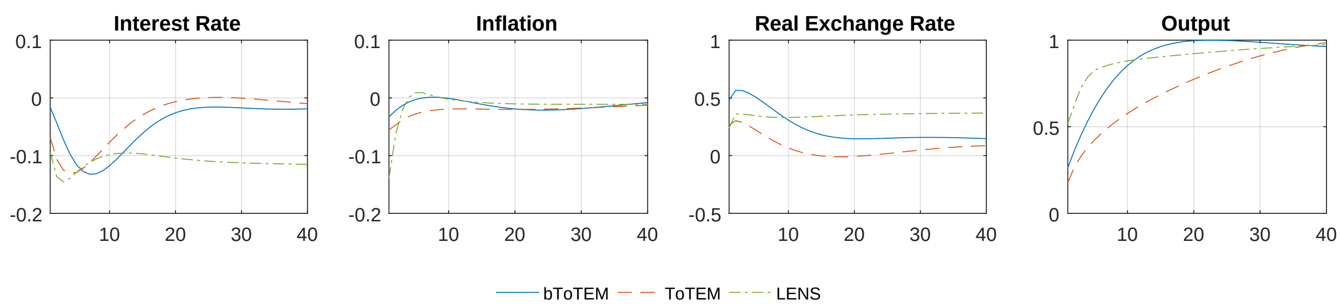

Figure 3: Impulse response functions: permanent productivity shock

In Figures 1–3, we plot impulse responses to three shocks in the bToTEM model, namely, an interest rate shock, a consumption demand shock, and a permanent productivity shock, respectively.[10] In Appendix E, we also plot impulse responses to foreign shocks, namely, ROW commodity price, demand, and interest rate shocks. In the figures, we report the response functions of four key model's variables: the nominal short-term interest rate, the rate of inflation for consumption goods and services, the real effective exchange rate, and the output gap. The responses are shown in percentage deviations from the steady state, except for the interest rate and the inflation rate, which are both shown in deviations from the steady state and expressed in annualized terms.

The responses we observe are typical for new Keynesian models. In Figure 1, a contractionary monetary policy shock leads to a decline in output through a decline in consumption. The uncovered interest rate

[10] Both, the ToTEM and LENS models, include more sources of uncertainty than the bToTEM model does, namely, 52 shocks in ToTEM and 98 shocks in LENS.

parity results in appreciation of the domestic currency. A reduction in the real marginal costs implies a lower price of consumption goods, and hence, lower inflation. In Figure 2, a negative shock to the discount factor increases consumption and decreases output. The interest rate that is determined by the Taylor rule increases, and the real exchange rate appreciates. In Figure 3, a permanent increase in productivity gives room for a higher potential output. The actual output gradually increases. Facing a negative output gap, the central bank lowers the interest rate according to the Taylor rule. As actual output reaches the new steady state level, the output gap closes, and the interest rate is back to the neutral rate. A lower interest rate leads to depreciation of the domestic currency because of the interest rate parity. Permanently higher productivity reduces input prices, leading to lower real marginal costs that are reflected in temporary lower inflation.

Our main finding is that our bToTEM model replicates the key properties of the full-scale ToTEM model remarkably well. Since ToTEM allows for multiple interest rates, different good prices, fiscal policies, etc., it has a richer structure than bToTEM. However, the variables that are the same in both models are described by essentially the same equations and therefore, have similar dynamics. One noticeable exception is the dynamics of inflation in response to a consumption demand shock; see Figure 2. In bToTEM, inflation reacts less on the impact, but decreases more slowly than in the other two models. To understand this difference between the two models, let us consider a linearized version of the Phillips curve, which is the same in bToTEM and ToTEM,

$$\hat{\pi}_t = (1-\theta)\,\gamma\omega\tilde{\phi}^{-1}\hat{\pi}_{t-1} + \beta\theta\tilde{\phi}^{-1}E\left[\hat{\pi}_{t+1}\right] + \tilde{\lambda}r\hat{m}c_t + \varepsilon_t^p, \tag{36}$$

where rmc_t is the real marginal cost; ε_t^p is a weighted average of the inflation target and the deviation of markup from the steady state; θ, γ, ω are the price stickiness parameters defined in Section 2.1; and $\tilde{\phi}$ and $\tilde{\lambda}$ are the parameters determined by equations $\tilde{\phi} = \theta + \omega\,(1-\theta)\,(1+\gamma\beta\theta)$ and $\tilde{\lambda} = (1-\omega)\,(1-\theta)\,(1-\beta\theta)\,\tilde{\phi}^{-1}$ (see equations (1.20)–(1.22) in Dorich et al., 2013). We observe that the difference in inflation dynamics is entirely attributed to the difference in the real marginal cost. In ToTEM, a consumption demand shock triggers a reallocation of inputs into the consumption production sector from the other four sectors. In the presence of adjustment costs, the reallocation raises the real marginal cost. In contrast, in bToTEM, there is one production sector and there are no input adjustment costs. Therefore, the responses and decays of the real marginal cost are less pronounced.

We also observe that the impulse responses of the ToTEM and bToTEM models are generally closer to one another than those produced by the ToTEM and LENS models, the two models of the Bank of Canada. This result is not surprising given that the bToTEM model is a scaled-down version of the ToTEM model, while LENS is a macroeconometric model constructed in a different way.[11] Consequently, our comparison results indicate that bToTEM provides an adequate framework for projection and policy analysis of the Canadian economy and that it can be used as a complement to the two models of the Bank of Canada.

4 Addressing the role of nonlinearities in the solution

Until the Great Recession, policymakers were not concerned with nonlinearities in their large-scale macroeconomic models. As Bullard (2013) pointed out, "... the idea that U.S. policymakers should worry about the nonlinearity of the Taylor-type rule and its implications is sometimes viewed as an amusing bit of theory without real ramifications. Linear models tell you everything you need to know. And so, from the denial point of view, we can stick with our linear models..." Also, Leahy (2013) argues: "Prior to the crisis, it was easier to defend the proposition that nonlinearities were unimportant than it was to defend the proposition that nonlinearities were essential for understanding macroeconomic dynamics." The question "How wrong

[11] LENS is not a general-equilibrium model that is derived from microfoundations and that is calibrated to the data, like ToTEM and bToTEM. It is a large-scale macroeconometric model composed of a set of equations whose coefficients are estimated from the data and are fixed for some period of time; see Gervais and Gosselin (2014) for a technical report about the LENS model.

could linearized solutions be?" became of interest to policymakers in light of the Great Recession and the recent zero lower bound (ZLB) crisis, and considerable efforts were dedicated to understand the role of nonlinearities in the implications of new Keynesian models. Why could nonlinearities matter for policy analysis? We distinguish three potential effects of nonlinearities on the properties of the solution compared with a plain linearization method:

*i). **(ELB)**. The ELB kink in the Taylor rule can induce kinks and nonlinearities in other variables of the model.*

*ii). **(Higher order terms)**. Higher order terms, neglected by linearization, can be quantitatively important for the properties of the solution.*

*iii). **(Solution domain)**. The quality of local (perturbation) solutions, constructed to be accurate in the steady state, can deteriorate when deviating from the steady state.*

To assess the quantitative importance of the above effects, we construct three numerical solutions to bToTEM, specifically: i) a first-order perturbation solution with occasionally binding constraints; ii) a second-order plain perturbation solution; iii) a nonlinear global solution.

4.1 A first-order perturbation-based solution with occasionally binding constraint

A plain first-order perturbation method is not suitable for approximating occasionally binding constraints like ZLB or ELB, however, there are perturbation-based methods that can approximate such constraints. In particular, IRIS software can handle occasionally binding constraints although it is limited to first-order approximation. This method allows us to construct policy projections conditional on alternative anticipated policy rate paths in linearized DSGE models; see Laséen and Svensson (2011) and Beneš (2015) on how this method deals with the inequality constraints; see also Holden (2016) for a related method. Dynare software cannot impose occasionally binding constraints itself, but there is an OccBin toolbox of Guerrieri and Iacoviello (2015) that allows imposing such constraints for first-order approximation. The method of Guerrieri and Iacoviello (2015) applies a first-order perturbation approach in a piecewise fashion to solve dynamic models with occasionally binding constraints. Thus, the first solution we report is a linear perturbation solution with occasionally binding constraints produced by IRIS (we also checked that the IRIS and OccBin toolboxes deliver identical results as pointed out in Guerrieri and Iacoviello, 2015).

4.2 A plain second-order perturbation solution

A plain second-order perturbation solution in a given class of economic models is given by

$$g(x,\sigma) \approx \underbrace{g(\bar{x},0) + g_x(\bar{x},0)(x-\bar{x})}_{\text{1st-order perturbation solution}} + \underbrace{\frac{1}{2}g_{xx}(\bar{x},0)(x-\bar{x})^2 + \frac{1}{2}g_{\sigma\sigma}(\bar{x},0)\sigma^2}_{\text{2nd-order terms}}, \tag{37}$$

where $g(x,\sigma)$ is a decision function to be approximates; x is a vector of endogenous and exogenous state variables; σ is a perturbation parameter that scales volatility; $(\bar{x},0)$ is a deterministic steady state; $g(\bar{x},0)$, $g_x(\bar{x},0)$ and $g_{xx}(\bar{x},0)$ are, respectively, steady state values, Jacobian and Hessian matrices of g; $(x-\bar{x})$ is a deviation from a steady state; and $(x-\bar{x})^2 \equiv (x-\bar{x}) \otimes (x-\bar{x})$ is a tensor product of the deviations. The first-order perturbation solution does not depend on the degree of volatility σ, i.e., $g_\sigma(\bar{x},0) = 0$; the term $g_{\sigma x}(\bar{x},0)$ is omitted as well because it is equal to zero; see Schmitt-Grohé and Uribe (2004).

Formula (37) shows the following: A second-order perturbation solution differs from a first-order solution by two terms: a constant term $\frac{1}{2}g^2_{\sigma\sigma}(\bar{x},0)\sigma^2$ (which we call an *uncertainty effect*), and a second-order term $\frac{1}{2}g_{xx}(\bar{x},0)(x-\bar{x})^2$ (which we call the *second-order effect*). To assess these effects, we report a plain second-order perturbation solution. We ignore ELB for second order perturbation solution.

4.3 A projection deep-learning global solution method

Perturbation solutions are local. They are constructed to be accurate in just one point – a deterministic steady state – and their quality can deteriorate when we deviate from the steady state. To assess the importance of the "solution domain" effect, we need to construct a global solution that approximates decision rules in a larger area of the state space.

But how can we construct a global solution to a model like bToTEM? First of all, the bToTEM model has much larger dimensionality than new Keynesian models studied with global methods in the literature, e.g., Judd et al. (2012), Gust et al. (2012), Fernández-Villaverde et al. (2012, 2015), Maliar and Maliar (2015), Boneva et al. (2016), Christiano et al. (2016), Aruoba et al. (2018), Coleman et al. (2018). For example, the model in Maliar and Maliar (2015), contains 8 state variables (6 exogenous and 2 endogenous ones) but bToTEM contains 21 state variables (6 exogenous and 15 endogenous ones). The difference between 8 and 21 state variables is immense: for example, if we discretize each state variable into just 10 grid points to construct a tensor product grid, we would have 10^8 and 10^{21} grid points, respectively, which implies a huge 10^{13}-times difference in evaluation costs. Clearly, conventional solution methods based on tensor product grids, such as conventional value function iteration, would be intractable in bToTEM!

High dimensionality is not the only challenge that bToTEM represents. The new Keynesian models that were studied in the literature by using global methods led to relatively simple systems of few equations that can typically be solved in a closed form, given future variables; e.g., Maliar and Maliar (2015). In turn, the open-economy bToTEM model produces a far more complex system of more than 30 equations that include both domestic and foreign variables. This system must be treated with a numerical solver in all grid points, as well as in all future states, inside the main iterative loop.

To solve bToTEM, we introduce a projection solution method that uses a combination of unsupervised and supervised (deep) learning techniques, specifically, we use clustering analysis for constructing the solution domain, and we use multilayer neural networks for approximating the decision functions. We complement our DL algorithm by numerical techniques that are designed to deal with large-scale applications, such as non-product monomial integration methods and derivative-free fixed-point iteration. Taken together, these techniques make our DL method tractable in problems with high dimensionality – dozens of state variables! Importantly, our solution method imposes ELB on nominal interest rates. In the remainder of this section, we outline the key ideas of these techniques; a detailed implementation of the DL method in the context of the bToTEM model is described in Appendix F.

4.3.1 Unsupervised machine learning: clustering analysis

To form a grid for constructing a global nonlinear solution to bToTEM, we use cluster grid analysis (CGA). Our solution method merges simulation and projection approaches, namely, it uses simulation techniques to identify a high probability area of the state space and it uses projection techniques to accurately solve the model in that area; see Maliar et al. (2011), Judd et al. (2011a, 2012) and Maliar and Maliar (2015) for a discussion of the related ergodic set methods and their applications.[12]

The CGA analysis can be understood by looking at a two-dimensional example in Figure 4.
In the first panel of the figure, we see a cloud of points that is obtained by stochastic simulation of an economic model: this cloud covers a high-probability area of the state space. The simulated cloud contains many redundant points that are located close to one another. The CGA method improves on pure stochastic simulation methods by eliminating the redundant points; specifically, it uses clustering analysis to replace a large cloud of simulated points with a smaller set of evenly spaced "representative"points. The remaining panels of the figure show how CGA constructs the representative points: it first combines simulated points into a set of clusters; it then distinguishes the centers of the clusters; and it finally uses the clusters' centers as a grid for constructing nonlinear solution.

[12] Maliar et al. (2011) and Judd et al. (2011a, 2012) introduced several numerical techniques in the context of solution methods for dynamic economic models that are extensively used in the recent machine learning literature (see, e.g., Goodfellow et al. (2016) for a review), including various regularized regressions, clustering analysis, epsilon-distinguishable sets, etc.

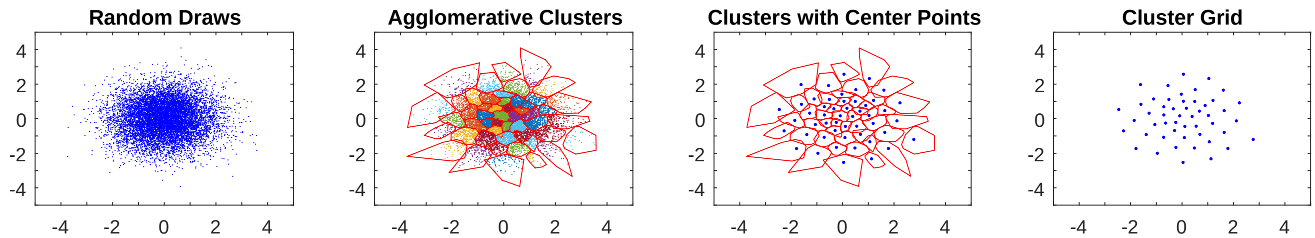

Figure 4: Construction of a cluster grid from simulated points

4.3.2 Supervised (deep) machine learning: multilayer neural network

The earlier version of our paper, Lepetyuk et al. (2017), approximated decision functions by second-degree polynomial functions. Those function were not flexible enough, and we ran into the problem of non-convergence. To achieve convergence, we scaled down the volatility of shocks but it also scaled down the effects of nonlinearities on the solutions. In the current version, we approximate decision functions by using neural networks, which we train by using the DL techniques; see Goodfellow et al. (2016) for a review of related computer science literature. The introduction of DL both increases accuracy and enhances convergence. As a result, our projection DL algorithm is capable of constructing fully nonlinear solutions under the empirically relevant volatility of shocks – this increases the role of nonlinearities in the solution.

The use of neural network in economic dynamics is dated back to Duffy and McNelis (2001) and McNelis (2005). While we were working on the paper, there appear several other papers that use neural networks for constructing their solutions, including Duarte (2018), Maliar et al. (2018, 2019), Villa and Valaitis (2019), Fernandez-Villaverde et al. (2019) and Azinović et al. (2019). There are important differences between this literature and our analysis. To be specific, Duarte (2018) and Fernandez-Villaverde et al. (2019) use supervised learning (neural networks) for approximating functions in continuous time models; in particular, the latter paper shows how neural network can be used to solve a challenging Krusell and Smith model. In turn, Maliar et al. (2018, 2019) show an approach that reformulates the entire economic model as an objective function of the deep-learning method (such as lifetime reward and residuals in the Euler and Bellman equations) and that trains the machine to optimize these objectives using the Google TensorFlow library. Azinović et al. (2019) use a similar method that minimizes Euler-equation residuals to solve a challenging high-dimensional life-cycle model. The paper that is closest to ours is a parameterized expectations algorithm with neural networks of Villa and Valaitis (2019) but our method is a projection and not a stochastic-simulation method: first, we use unsupervised learning (clustering) to construct a fixed grid for computing the solution instead of simulation; and second, we use accurate deterministic integration for evaluating expectation functions instead of less accurate Monte Carlo integration; see Maliar and Maliar (2014) for a comparison of projection and simulation methods. As a result, our DL method produces highly accurate solutions for the challenging bToTEM model.

In Figure 5, we show the neural network with two layers that we use: a hidden layer and an output layer. The hidden layer takes as inputs normalized input variables x_i. An activated neuron of this layer is a result of applying an activation function to the neuron-specific weighted sum of the inputs plus a bias. The output layer takes as inputs the activated neurons of the hidden layer and delivers the normalized output variables of the network. In the figure, τ_1 and τ_2 stand for activation functions of the hidden and output layers, respectively.

A straightforward use of neutral networks for constructing decision functions would take as an input the vector of state variables. We choose instead as an input to our neural network a quadratic base of the state variables. For 21 state variables of the model, the quadratic base consists of 252 variables plus a constant term. This approach allows us to reduce the degree of nonlinearity explicitly attributed to the neural network and to limit the network to one hidden layer. The reduction of nonlinearity of the network is useful for convergence of an iterative solution algorithm.

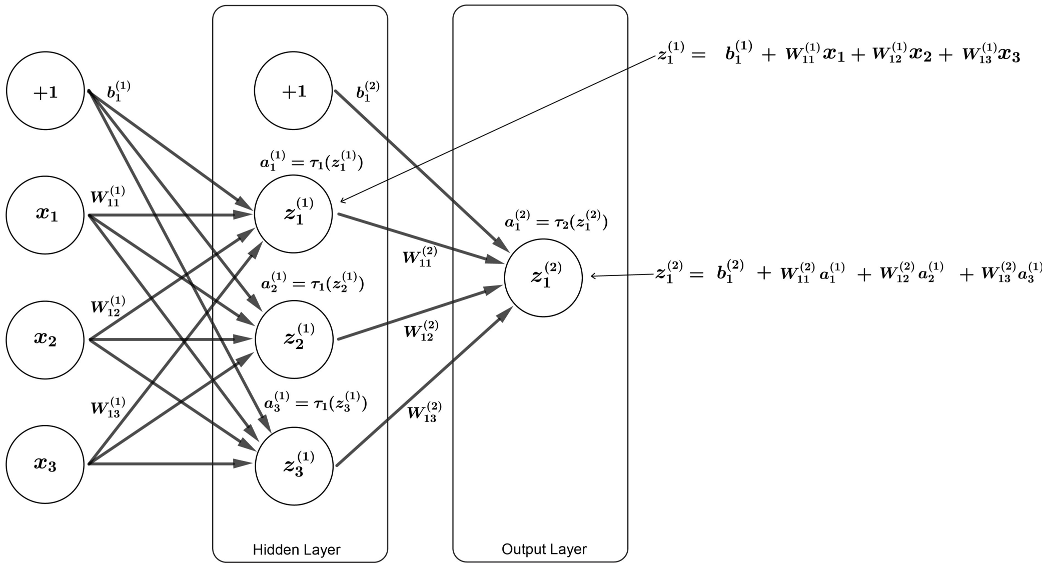

Figure 5: Three layer neural network

All inputs and outputs of the neural network are normalized by a linear transformation to lie within an interval $[-1; 1]$. In the hidden layer of the network, a normalized input vector x is transformed to an activated neuron vector $a^{(1)}$ by a symmetric sigmoid activation function. For a neuron i, the transformation is as follows:

$$a_i^{(1)} = \text{tansig}\left(z_i^{(1)}\right) = \text{tansig}(b_i^{(1)} + W_i^{(1)}x),$$

where $z_i^{(1)}$ is a nonactivated neuron i of the hidden layer, $b_i^{(1)}$ is a scalar that captures a bias, $W_i^{(1)}$ is a vector of weights, and $\text{tansig}(x) = 2/\left[1 + \exp(-2x)\right] - 1$ is an activation function τ_1. In the output layer of the network, the neurons are transformed to normalized output variables by a linear function, i.e., τ_2 is linear. For output, the transformation is given by

$$y_i = z_i^{(2)} = b_i^{(2)} + W_i^{(2)}a^{(1)},$$

where $z_i^{(2)}$ is output (both activated and nonactivated as the activation function is linear), $a^{(1)}$ is a vector of activated neurons of the hidden layer, $b_i^{(2)}$ is a scalar capturing a bias, and $W_i^{(2)}$ is a vector of weights.

In our implementation, we use 11 neurons in both network layers to match the number of intertemporal choice variables. For this number of neurons, the neutral network includes 2,915 coefficients. There are 2,783 coefficients in the hidden layer, and there are 132 coefficients in the output layer.

5 Is the Canadian ELB crisis imported from abroad?

In the U.S. and European countries, the Great Recession and ZLB episodes were caused by the 2008 financial crisis. In contrast, Canada did not experience any significant financial crisis or economic slowdown at the beginning of the Great Recession. Nonetheless, Canada also ended up reaching an ELB on nominal interest rates and remained there during the 2009–2010 period. To be specific, the Bank of Canada targeted the overnight interest rate at 0.25 percent annually, which at that time was viewed by the Bank to be a lower bound on the nominal interest rate.

What factors led the Canadian economy to the ELB crisis? In the first experiment, we argue that the recession spread to Canada via the rest of the world, primarily from the U.S., which is the main Canadian trade partner (around 75 percent of Canadian exports go to the U.S.). The Canadian economy experienced a huge (16 percent) drop in exports in the beginning of the Great Recession; see a speech by Boivin (2011), a former Deputy Governor of the Bank of Canada. Using bToTEM, we find that a negative ROW shock of such magnitude is sufficient to originate a prolonged ELB episode in the Canadian economy.

5.1 Calibrating exogenous ROW shocks from ToTEM

An important question is how to realistically calibrate the behavior of the ROW sector in the bToTEM model since a foreign financial crisis affects not just foreign demand but also foreign prices and foreign interest rates. Our methodology combines the analysis of bToTEM and ToTEM. Namely, we use ToTEM to produce impulse responses for three foreign variables: a ROW interest rate, ROW commodity price and ROW output, and we use these three ToTEM variables as exogenous shocks in the bToTEM model; these shocks are shown in Figure 6. In ToTEM, a negative shock in the ROW sector has three effects: the world

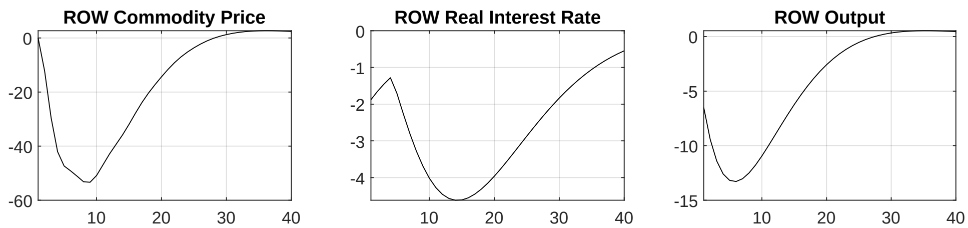

Figure 6: Exogenous ROW shocks

demand goes down, output falls, and the ROW commodity price reduces. Since the monetary authority in the ROW model is assumed to follow a Taylor rule, the ROW nominal interest rate goes down as well. The size of the considered ROW shock in ToTEM is such that its output declines by 7 percent on the impact of shock, and it declines by 12 percent at the peak – these numbers are consistent with the magnitudes of foreign shocks experienced by the Canadian economy during the Great Recession.

5.2 Generating the ELB episode

Figure 7 displays the simulated time series for the key model variables under the given behavior of the ROW sector imported from ToTEM. Here, the ELB on the nominal interest rate is set at 2 percentage points below the deterministic steady state of the nominal interest rate. All the variables are reported in percentage deviations from the deterministic steady state, except of inflation and the interest rates that are shown in annualized deviations from the deterministic steady state. We assume that initially, the domestic interest rate in bToTEM is slightly below the deterministic steady state, namely, by 1 percent, which makes it is easier to reach the ELB on the nominal interest rate.[13]

We plot three different solutions, namely, a first-order perturbation solution produced by IRIS (or OccBin) with the ELB imposed; a plain second-order perturbation solution produced by Dynare without imposing ELB; and a DL solution with the ELB imposed.[14] As we see, the three solutions look very similar in the figure.

To check the accuracy of numerical solutions, we compute unit-free residuals in the model's equation along the simulation path; see Appendix G for details of our accuracy assessment. As expected, the global solution method is the most accurate. The least accurate first-order perturbation methods can produce residuals of order $10^{-1.43} \approx 3.7$ percent, while the DL method produces residuals which are a half order of magnitude lower, namely, equal to $10^{-2.09} \approx 0.8$ percent. (The accuracy results are similar on a stochastic

[13] The natural yearly rate of interest in bToTEM is calibrated to 3 percent as in ToTEM. This value is chosen to represent the long-run historical average of the natural rate of interest in the Canadian economy. However, the current natural rate of interest in Canada is considerably lower. Setting the initial interest rate below the steady state is a way to account for the current low interest rate.

[14] We use pruning to simulate the second-order perturbation solution; see Andreasen et al. (2013).

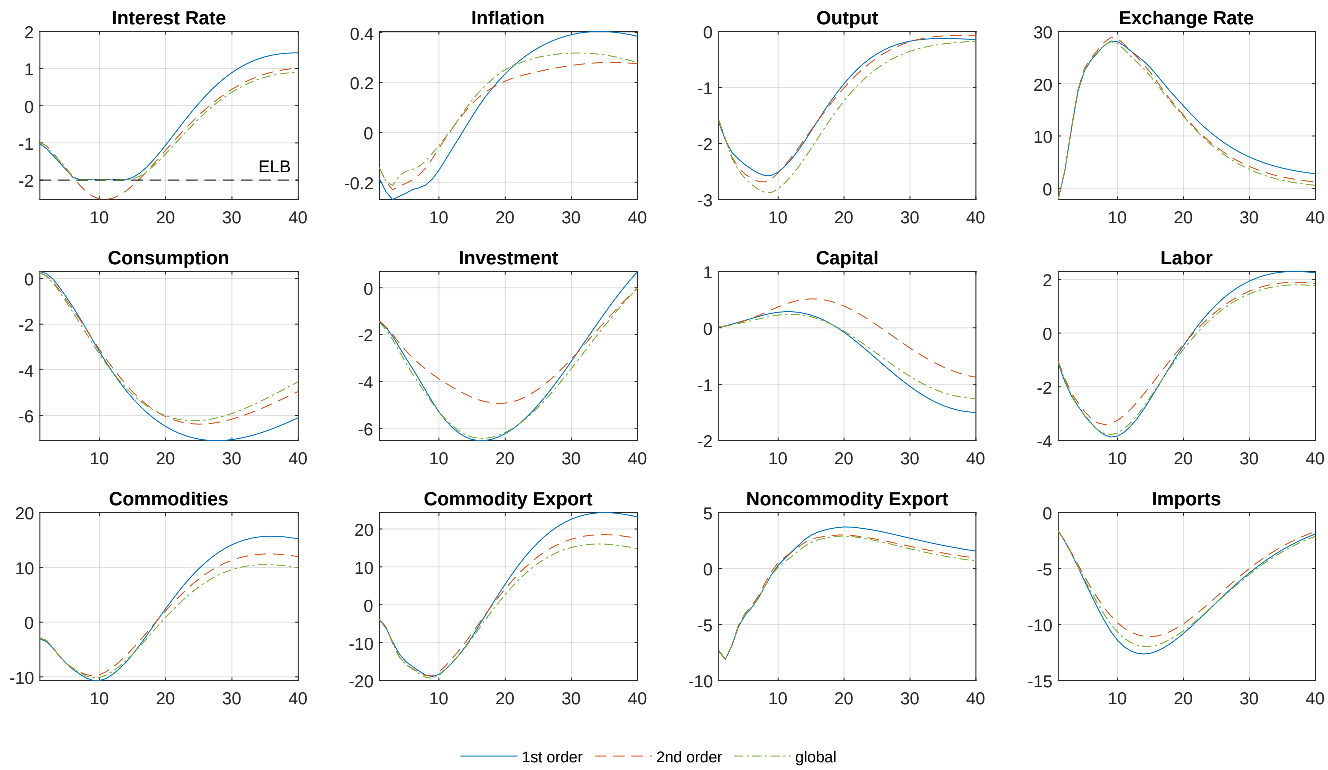

Figure 7: Responses of linear perturbation, quadratic perturbation, and global ML solutions to ROW shocks

simulation). Given that all three numerical solutions look similar, we conclude that numerical errors of these magnitudes do not affect the qualitative implications of the model in this experiment.

5.3 Understanding the ELB episode in Canada: a contagion mechanism

In our experiment in Figure 7, the nominal interest rate reaches the ELB and remains there during eight quarters, which corresponds to what actually happened in the Canadian economy during the 2009–2010 period. Our analysis suggests that a contamination-style mechanism accounts for this ELB episode in the Canadian economy. Under the considered scenario of negative ROW shocks, there are three foreign variables that decline during the crisis, namely, foreign output, the foreign interest rate, and the world commodity price; see Figure 6. The immediate consequence of these shocks for the domestic economy is a sharp decline in commodity and noncommodity exports in the Canadian economy. There are significantly fewer commodities extracted due to a huge decline in commodity prices and as a consequence, domestic output starts declining. The central bank responds by lowering the interest rate to stimulate the economy but the magnitude of shocks is so large that the bank reaches the lower bound on the nominal interest rate by six quarters. Without unconventional monetary tools, the interest rate stays at the lowest value for more than two years until the foreign economy sufficiently recovers. All three numerical methods considered deliver the same qualitative predictions about the ELB episode.

There is related recent literature on the transmission of liquidity trap from one country to another.[15]

[15] There is also earlier literature that analyzes the effects of foreign shocks on a domestic economy over the business cycle; see e.g., Backus et al. (1992). Schmitt-Grohé (1998) finds that variations in export demand are more important for explaining the business cycle behavior of Canadian aggregate variables than variations in financial markets. Lubik and Schorfheide (2005) build a new Keynesian model with two countries – United States and Euro area – and find asymmetric transmissions of monetary, supply and demand shocks. See also Fernández, Schmitt-Grohé, Uribe (2017) for recent evidence on the importance of terms of trade shocks over the business cycle.

Fujiwara (2010) considers a small open economy, calibrated to the Japanese economy, and shows analytically that the multiplier of an export demand shock is small if an economy is not hit by ZLB but increases by a factor of 100 if such an economy is at the binding ZLB constraint. Jeanne (2010) uses a two-country model to show that a negative demand shock in one country may push the other country to zero nominal interest rates. Bodenstein et al. (2016) find that if a domestic economy (U.S.) is not at ZLB, negative foreign shocks have negligible effects on the U.S., but if the U.S. is previously hit by ZLB, the effects of such shocks are substantially amplified. Other papers that share similar themes include Cook and Devereux (2011, 2013, 2016), and Corsetti et al.(2016) among others. Finally, Devereux (2014), Caballero et al. (2016), and Eggertsson et al. (2016) analyze how liquidity traps spread across the world by emphasizing the role of capital flows.

5.4 ELB is easy to generate in open-economy models unlike in closed-economy models

Surprisingly, we find that it is fairly easy to generate prolonged ELB episodes in an open-economy setting, while it is quite difficult to produce such episodes in closed-economy models. In particular, Chung, Laforte, Reifschneider and Williams (2012) find that standard structural models (FRB/US, EDO, Smets and Wouter (2007)) deliver very low probability of hitting the ZLB. Maliar and Maliar (2015) generate the ZLB episodes by assuming large preference shocks affecting the marginal rate of substitution between consumption and leisure. Aruoba et al. (2018) augment the simulated series from the model to include historical data from the U.S. economy in order to obtain realistic spells at the ZLB. Fernández-Villaverde et al. (2012, 2015) argue that within a standard new Keynesian model, it is impossible to generate long ZLB spells with modest drops in consumption, which were observed during the recent crises; they suggest that the only way to get around this result is to introduce wedges into the Euler equation. Also, Christiano et al. (2015) emphasize the importance of such shocks as a consumption wedge (a perturbation governing the accumulation of the risk-free asset), a financial wedge (a perturbation for optimal capital accumulation), a TFP shock, and a government consumption shock. Thus, our contamination-style motive for the ELB crisis in the open-economy model of the Canadian economy differs from those proposed in the literature for closed-economy models.

5.5 The irrelevance of the ZLB hypothesis

The role of ELB is modest in our first experiment: all three solutions predict nearly the same magnitude and duration of the ELB crisis. This result suggests that all three kinds of nonlinearities described in Section 4 are quantitatively unimportant: First, the presence of an active ELB does not significantly affect the behavior of other variables; to put it simply, if the Bank of Canada just used a plain first-order perturbation method for analyzing ToTEM, either ignoring ELB entirely or chopping the interest rate at ELB in simulation, it would not be terribly wrong. Furthermore, a comparison of the first- and second-order perturbation solutions suggests that the role of second-order terms is also relatively minor. Finally, given that our global DL solution is relatively close to perturbation solutions, we conclude that the quality of perturbation solutions does not dramatically deteriorate away from the steady state.

The related literature focuses almost exclusively on nonlinearities in the new Keynesian models resulting from ZLB or ELB.[16] The findings of this literature are mixed. Several papers find that ZLB is quantitatively important in the context of stylized new Keynesian models with Calvo pricing, in particular, Maliar and Maliar (2015) argue that first- and second-order perturbation solutions understate the severity and duration of the ZLB crisis; Fernández-Villaverde et al. (2015) show that the nonlinearities start playing an important role when ZLB is binding, affecting the expected duration of spells, fiscal multipliers, as well as the trade-off between spells and drops in consumption; and Aruoba et al. (2018) show that nonlinearities in their new Keynesian model can explain the differential experience of the U.S. and Japan by allowing

[16] One exception is Judd et al. (2017), who demonstrate that approximation errors in linear and quadratic perturbation solutions can reach hundreds percent under empirically relevant calibrations of new Keynesian models, even if the economy is not at the ZLB.

for nonfundamental shocks (sunspots). Furthermore, in the model with Rotemberg pricing, Boneva et al. (2016) find that linearization considerably distorts the interaction between the ZLB and the agents' decision rules, in particular, those for labor supply.[17]

There is a growing stream of the literature advocating the ZLB irrelevance hypothesis which argues that the impact of binding ZLB on the economy's performance is insignificant. First, it includes empirical literature that estimated impulse response functions from the data and find that macroeconomic variables were hardly affected by active ZLB; see Debortoli et al. (2019) for a review of the literature. Furthermore, it includes literature that solves new Keynesian models and finds that ZLB is quantitatively unimportant. In particular, Christiano et al. (2016) study a stable-under-learning rational expectation equilibrium in a simple nonlinear model with Calvo pricing; they find that a linearized model inherits the key properties of the nonlinear model for fiscal policy at ZLB, predicting similar government spending multipliers and output drops. Furthermore, Eggertsson and Singh (2016) derive a closed-form nonlinear solution to a simple, two-equation new Keynesian model. They report negligible differences between the exact and linearized solutions when they look at the effects of fiscal policy at ZLB.

Thus, our findings from the bToTEM model are in line with the literature that did not find important effects of nonlinearities. Clearly, a modest role of nonlinearity in our first experiment is not generic but a numerical result that is valid just for our specific set of assumptions and calibration procedure. It happens because the probability of ELB is relatively low, as well as its cost, so the possibility of hitting ELB does not considerably affect the decisions of the agents. We can increase the importance of ELB by increasing the volatility of shocks (still within a reasonable range) or by modifying some model's assumptions (one such modification is shown in Section 7). However, demonstrating the importance of nonlinearities and ELB was not the goal of our analysis. We meticulously calibrate the bToTEM model to reproduce the Canadian data, trying to make it as close as possible to the full-scale ToTEM model; and under our calibration, nonlinearities proved to be quantitatively unimportant, including ELB. Our negative result does not mean that the Great Recession was unimportant but that the main mechanism of the recession was something else rather than a binding ELB constraint.

6 Preventing the recurrence of the ELB crisis

For the last 25 years, the Bank of Canada has adhered to the inflation targeting, but every three to five years it revises its inflation-control target level. The last revision happened in 2016: the Bank of Canada considered the possibility of increasing the inflation target, but eventually it reached the decision to keep it at a 2 percent level for the next five years. A higher inflation target reduces the probability of reaching ELB, but it also has certain costs for the Canadian economy; see Kryvtsov and Mendes (2015) for some considerations that may have influenced the decision of the Bank of Canada. In our second experiment, we use a simulation of bToTEM to assess the impact of a hypothetical transition from a 2 to 3 percent inflation target on the Canadian economy.

6.1 A 3 percent inflation target would have prevented the 2009-2010 ELB episode

In the new Keynesian models like ToTEM and bToTEM, the ELB or ZLB episodes can be prevented by choosing a sufficiently large inflation target. For example, in Figure 8 we show that if the Bank of Canada had the inflation target of 3 percent instead of 2 percent, the ELB would never be reached in bToTEM under any solution method in our first experiment.

The idea of using an inflation target as a policy instrument for dealing with ZLB episodes dates back at least to Summers (1991) and Fischer (1996), who suggest to use an inflation target in the range of 1 to 3 percent if the economy hits ZLB. Krugman (1998) proposes to use a 4 percent inflation target in the Japanese economy to deal with persisting deflation. In light of the Great Recession, Blanchard,

[17] See also Gust et al. (2012) for related evidence from estimation of a nonlinear new Keynesian model using a maximum-likelihood approach.

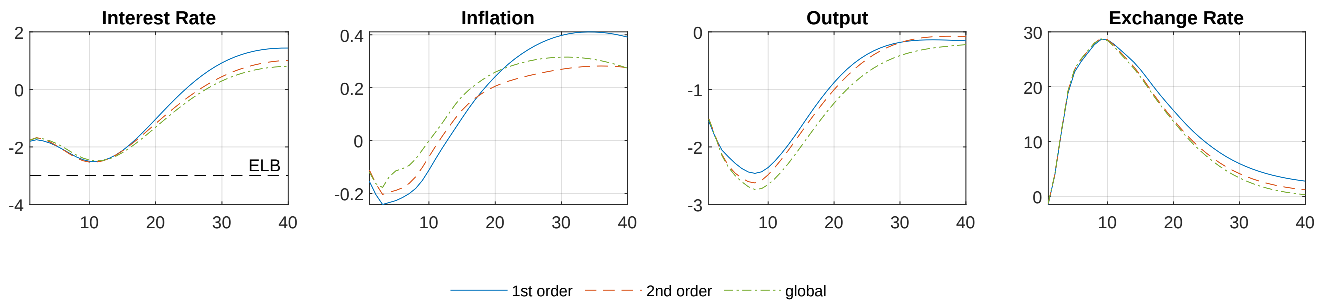

Figure 8: Linear perturbation, quadratic perturbation, and global ML solutions under the inflation target of 3 percent (in deviations from the deterministic 3%-inflation-target steady state)

Dell'Arriccia and Mauro (2010) argue that adopting a 4 percent inflation target in the U.S. can help avoid the ZLB crisis; see also Williams (2009) and Ball (2013) for related proposals. For the Canadian economy, Dorich et al. (2018) estimate the effects of the increases in the inflation target to 3 percent (or 4 percent) and they find that it decreases the probability of hitting the ELB from 8 percent to 4 percent (or 2 percent).

However, a higher inflation target has also certain costs, in particular, it tends to decrease average output, to increase costs of price dispersion, to raise the economy's volatility and to produce unstable expectations dynamics; see Ascari and Sbordone (2014) for a discussion. Coibion et al. (2012) perform a careful assessment of the costs and benefits of higher inflation: they find that for plausible calibrations of the model and realistic frequencies of the ZLB episodes (of eight quarters), the optimal inflation rate in the U.S. economy must be less than 2 percent: the cost of ZLB in their analysis is relatively low because the probability of hitting ZLB is relatively low.

6.2 Modeling a transition from a 2 percent to 3 percent inflation target

We use the bToTEM model to study a transition of the Canadian economy after a hypothetical increase in the inflation target from 2 percent to 3 percent, a possibility that was recently evaluated by the Bank of Canada. We implement a change in the inflation target by maintaining the real neutral interest at the same level of 3 percent. We thus simultaneously adjust the nominal interest rate target level from about 5 to 6 percent.

In our baseline experiment, the initial condition corresponds to the deterministic steady state of the Canadian economy with an old inflation target of 2 percent. We then recompute the solution under the new inflation target of 3 percent, and we simulate the transition path from the old to the new steady states. In all cases, we assume no shocks over the transition path.

6.3 Dramatic differences between local and global solutions

In Figure 9, we plot a simulation for the first- and second-order perturbation solutions, as well as global DL solutions. Here, the level "0" corresponds to the initial deterministic steady state of 2 percent and all the variables are given in deviations from that initial steady state; the level "1" in the figure for inflation means 3 percent and "1" in the figure for interest rate means a new nominal interest rate target of 6 percent.

In this experiment, the three solutions look dramatically different. The first-order perturbation solution behaves in a way that is typical for new Keynesian models and that agrees with our intuition and common sense. The change in the inflation target almost instantaneously translates into an increase in the inflation rate. Inflation reacts so rapidly because in our sticky-price economy, the non-optimizing firms set their price according to the inflation target, which is instantaneously changed from 2 percent to 3 percent. Following the Taylor rule with persistence, the interest rate reaches the new steady state within a couple of years. During the transition to this new steady state, the interest rate is below the new steady states and therefore

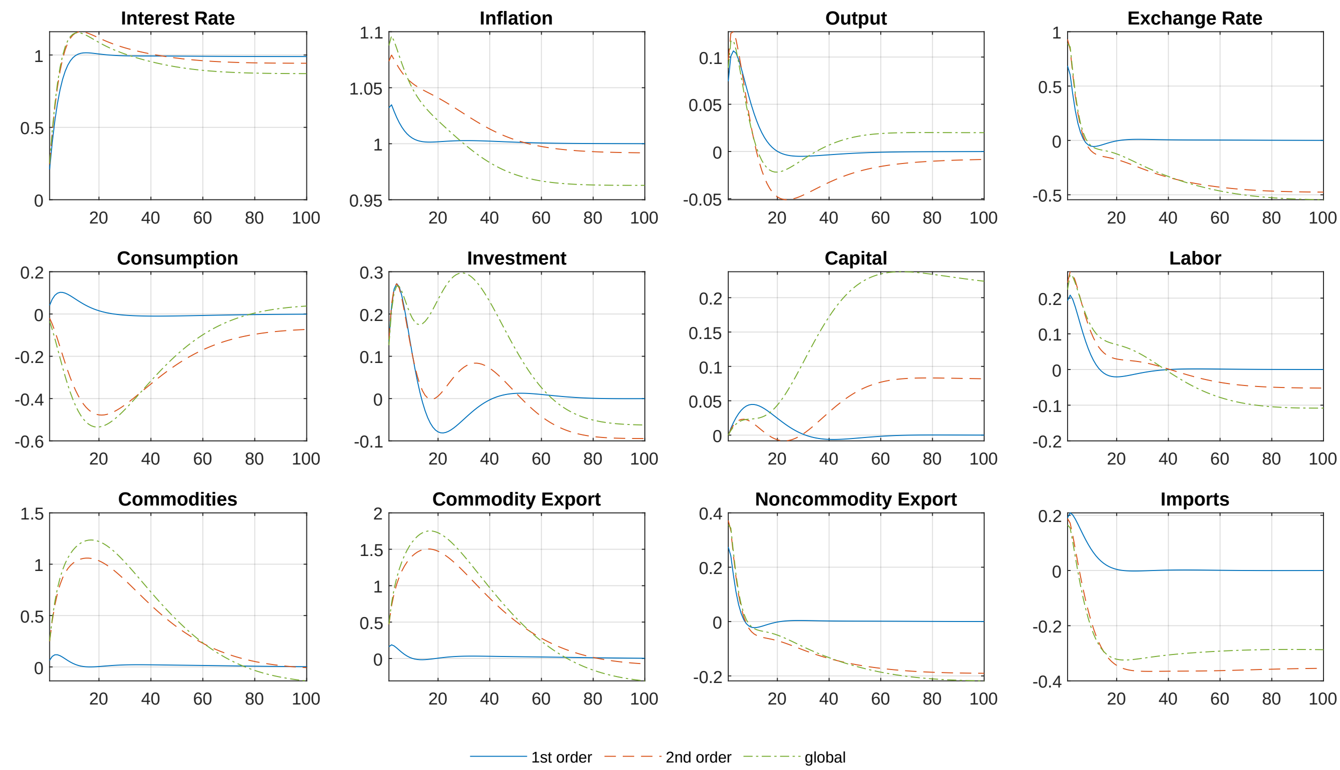

Figure 9: A transition from a 2 percent to 3 percent inflation target under linear, quadratic, and global ML solutions (in deviations from the deterministic 2%-inflation-target steady state)

provides a monetary stimulus. The stimulus is reflected in higher investment, output, consumption and capital under all solutions.

In contrast, the simulation of second-order and global nonlinear solutions looks odd, in particular, consumption and investment produce wiggles and even go down. However, these implications of the second-order perturbation and global solutions are puzzling and seem to point to the importance of some nonlinear effects, but we will be able to resolve this puzzle below.

6.4 Understanding the impact of uncertainty on the steady state

Let us recall the three effects of nonlinearities discussed in Section 4. Since ELB is not binding in this experiment, the first- and second-order perturbation solutions in (37) can differ either because of the uncertainty effect represented by a constant term $\frac{1}{2}g^2_{\sigma\sigma}(\bar{x},0)\,\sigma^2$ or because of the second-order effects represented by a quadratic term $\frac{1}{2}g_{xx}(\bar{x},0)\,(x-\bar{x})^2$. Global solutions can also differ from perturbation solutions because their coefficients are constructed on a larger solution domain.

The uncertainty effect means that linear and nonlinear models have different steady states: In absence of shocks, a linear model converges to the deterministic steady state, while a nonlinear model converges to the so-called stochastic steady state that depends on a degree of volatility σ.

The uncertainty effect is well appreciated from Figure 9: both second-order perturbation and global DL solutions converge not to the deterministic steady state but to some other levels. In particular, in the nonlinear case, the interest rate does not increase by exactly 1 as in the linear case, but somewhat less. This means that the central bank does not get the same inflation rate as it targets by the Taylor rule in the nonlinear case (if the initial condition is a deterministic steady state); see Hills et al. (2016) for an estimation of such deflationary bias in the U.S.

In turn, the term $\frac{1}{2}g_{xx}(\bar{x},0)\,(x-\bar{x})^2$ captures second-order effects that are ignored by linear solutions, in particular, those associated with wage and price dispersions (in linear solutions, such dispersions are equal to zero). To assess the relative importance of these and other similar second-order effects, in Figure 10 we simulate a second-order perturbation solution by using a stochastic steady state as an initial condition instead of the deterministic one.

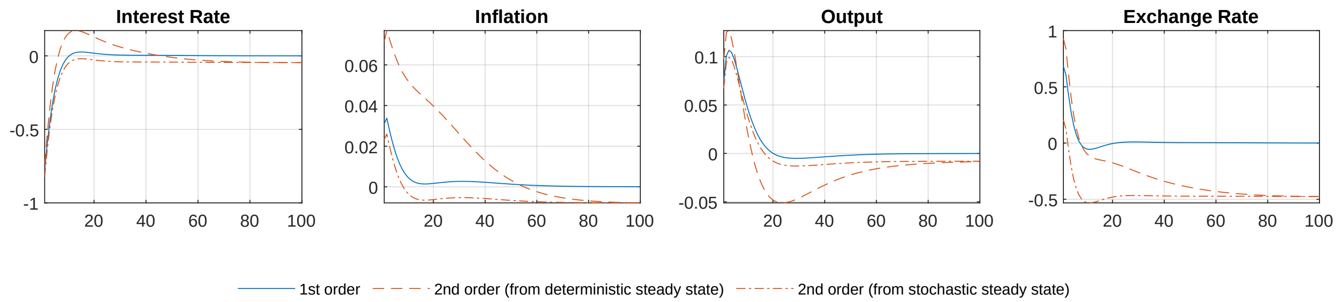

Figure 10: A transition from a 2 percent to 3 percent inflation target under linear and two quadratic solutions one of which starts from the deterministic steady state and the other starts from the stochastic steady state (in deviations from the deterministic 3%-inflation-target steady state).

Surprisingly, once the initial condition is adjusted, the second-order effects disappear! In Figure 10, the first-order and alternative second-order perturbation solutions are visually indistinguishable, up to a constant term that shifts the second-order solution relative to the first-order solutions. Now, we realize that the puzzling behavior of nonlinear solutions in Figure 9 such as wiggly consumption, investment and capital along the transition happens simply because nonlinear solutions are effectively confronted with two transitions: one is a transition to a new inflation target and the other is a transition from the deterministic to their own stochastic steady state. Adjusting the initial condition removes the second transition and makes the nonlinear solutions meaningful. This means that second-order effects associated with the wage and price dispersion play a relatively minor role in second-order perturbation solutions.

We next perform a similar experiment with the global DL solution by constructing an alternative simulation that starts from the stochastic steady state of the DL global solution; see Figure 11.

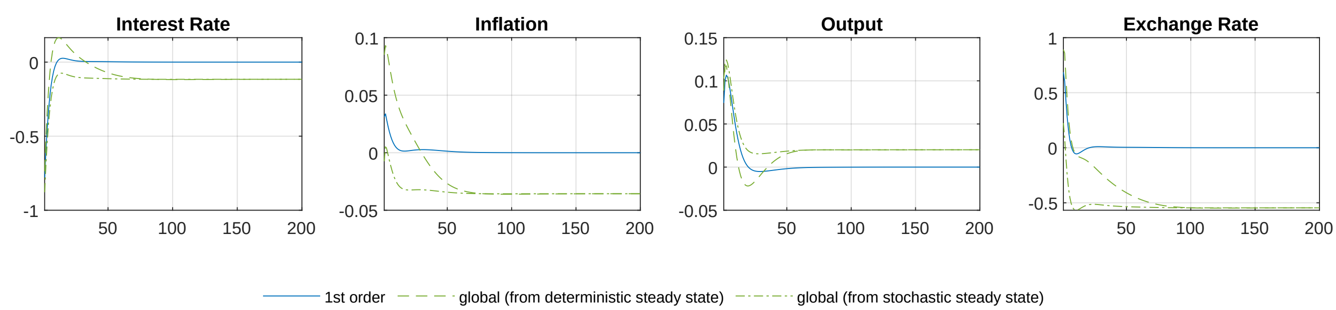

Figure 11: A transition from a 2 percent to 3 percent inflation target under linear and two global ML solutions one of which starts from the deterministic steady state and the other starts from the stochastic steady state (in deviations from the deterministic 3%-inflation-target steady state).

The results for the global DL solution are similar to those of the second-order perturbation solution. Here, we also observe an important uncertainty effects on the steady state but once we make an adjustment for differing steady states, the local and global solutions become qualitatively similar. However, we also observe a visible "solution domain" effect that amplifies the role of higher order nonlinearity terms. In particular, the long-term changes of the interest and inflation rates are visibly larger for the global DL solution than for the second-order perturbation solution. The consequence of a larger stimulus is that the effect of the inflation-target change on output is positive for the global DL solution, while it was negative for the second-order perturbation solution.

An important lesson from our analysis is that a coherent simulation of numerical solutions requires us to start each solution from its own steady state (or from the same relative distance from the steady state).

7 The role of the closing condition

Our previous analysis seems to suggest that nonlinearities, including ELB, play a relatively minor role in the bToTEM performance (provided that we account for differing steady states). In this section, we show that a relatively small change in the model's assumptions can change this conclusion, such as a variation in the closing condition used to induce stationarity in open-economy models.

7.1 Closing condition matters in the bToTEM model

Let us replace the linear closing condition (35) that ensures stationarity in the bToTEM model with a similar closing condition in an exponential form, as is used in Schmitt-Grohé and Uribe (2003):

$$\kappa_t^f = \varsigma \left[\exp\left(\bar{b}^f - b_t^f \right) - 1 \right]. \tag{38}$$

Let us re-visit our first experiment, in which the Canadian economy experiences three shocks to the ROW variables. Recall that under our benchmark linear closing condition in Figure 7, all three solutions looked very similar. However, Figure 12 provided below shows that a second-order perturbation solution looks very different after we change the closing condition to the exponential one in (38).

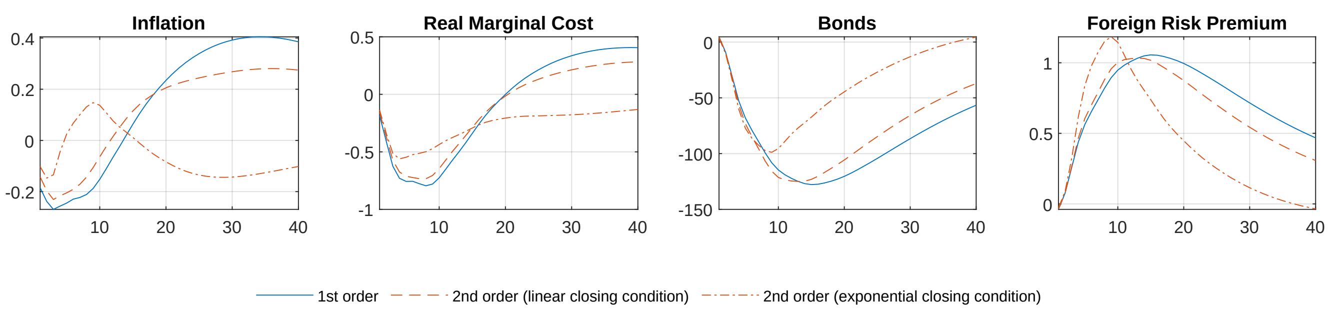

Figure 12: Responses of linear perturbation, quadratic perturbation and alternative quadratic solutions to ROW shocks.

Now, the risk premium has a faster and sharper increase and decline because of the changes in the foreign bonds. The change in risk premium dynamics affects the exchange rate via the uncovered interest rate parity condition (31) (it pre-multiplies the foreign interest rate). In turn, the exchange rate affects the prices of two out of four inputs in the production function, which eventually affects the real marginal cost and inflation.

Why are closing-condition nonlinearities manifested in bToTEM? The linear closing condition (35) does not have second-order terms and is equal to itself $\kappa_t^f = \varsigma \left(\bar{b}^f - b_t^f \right)$, while the exponential closing condition (38) does have such terms and is given by $\kappa_t^f \approx \varsigma \left(\bar{b}^f - b_t^f \right) + \frac{1}{2}\varsigma \left(\bar{b}^f - b_t^f \right)^2$. Precisely, the second-order term $\frac{1}{2}\varsigma \left(\bar{b}^f - b_t^f \right)^2$ accounts for a visibly large difference between the first- and second-order perturbation solutions in Figure 12.

7.2 But closing condition does not matter in Schmitt-Grohé and Uribe (2003)

The importance of stationarity condition in the bToTEM model is surprising, given a well-known finding of Schmitt-Grohé and Uribe (2003) that closing conditions does not matter for implications of open-economy models. However, we shall recall that their analysis focuses exclusively on first-order solutions. For such solutions, the exponential stationarity condition (38) is given by $\kappa_t^f \approx \varsigma \left(\bar{b}^f - b_t^f\right)$, which exactly coincides with the linear closing condition (35). Thus, the analysis of Schmitt-Grohé and Uribe (2003) would treat the two alternative closing conditions as exactly identical and it would not discover the importance of nonlinearity effects associated with the closing condition, even if such effects were present.

Schmitt-Grohé and Uribe (2003) consider a small open economy model with incomplete asset markets, which is parameterized and calibrated as in Mendoza (1991). We ask whether the conclusions of the analysis in Schmitt-Grohé and Uribe (2003) will change if instead of linearization, we solve their model by using high-order perturbation and global nonlinear solution methods? We find that the answer is "No", namely, we find that nonlinearity effects associated with the closing condition play a very minor role in the predictions of Schmitt-Grohé and Uribe's (2003) model. To illustrate our finding, in Figure 13, we show responses of selected variables to a 10-percent productivity shock in their model. To make the nonlinearity

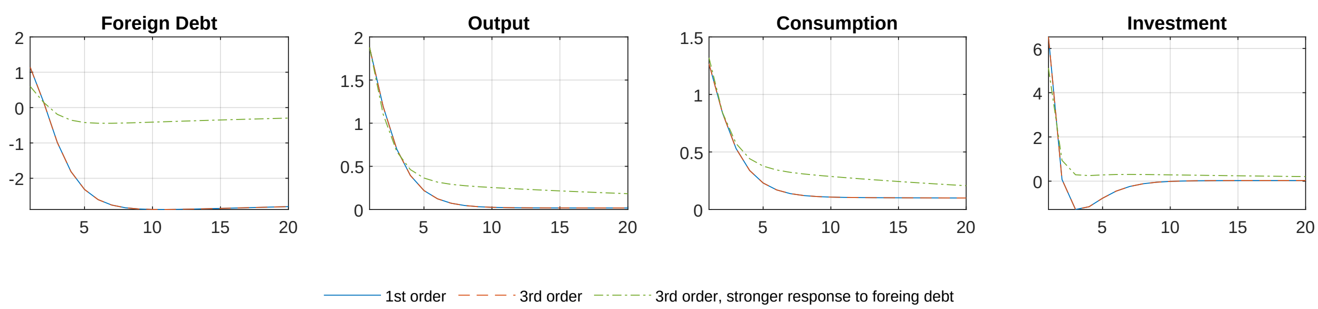

Figure 13: Responses to a productivity shock in the model of Schmitt-Grohé and Uribe (2003).

effects most visible, we report more nonlinear third-order perturbation solution instead of the second-order one. Still, under their benchmark parameterization, the nonlinearity effects are practically absent, so that the linear and nonlinear solutions are essentially identical. We finally perform a sensitivity experiment in which we increase the value of the parameter ς by 100; this parameter controls the importance of higher order terms in the model of Schmitt-Grohé and Uribe (2003). In that extreme case, we observe that the responses of variables are dampened in the third-order solution relatively to first-order solutions, especially, the response of the foreign debt. But even in that case, the dynamics of the model do not change qualitatively and the difference between first- and third-order solutions is relatively small.

Thus, our findings do not invalidate the insights from the analysis of Schmitt-Grohé and Uribe (2003). However, they suggest that the regularities observed in one model do not necessarily carry over to other models, in particular, to more complex central banking models. "Innocent" assumptions that we do not expect to play a role in the properties of the solution can lead to important nonlinearity effects. Our analysis suggests that central banks must be systematically checking the robustness of their linear solutions to potentially important effects of nonlinearities.

8 Conclusion

This paper tells a tale of the Canadian ELB experience during the Great Recession: We demonstrate that a direct impact on the foreign trade was a quantitatively important transmission channel through which the contagion of the Great Recession spread to Canada from the rest of the world. There is a popular saying "When the U.S. sneezes, Canada catches a cold". But this time it went the other way around: it was that the U.S. which caught the (subprime crisis) cold, and it was Canada which sneezed.

Our tale builds around a carefully designed ToTEM model, which is meticulously calibrated to reproduce the key observations on the Canadian economy, as well as the impulse responses of ToTEM. The bToTEM model is capable of generating the ELB episode under the rest-of-the-world shocks that are backed up from the actual data by using the full-scale ToTEM model.

Large-scale central banking models like ToTEM and bToTEM are routinely used for similar policy experiments but their analysis is limited to linear approximations. Our novel DL algorithm combines supervised and unsupervised learning in a way that enables us to construct accurate global fully nonlinear solutions to a central banking model with the degrees of nonlinearities and the size of the state space that has never been studied before.

What is the value added of DL for telling the Canadian ELB tale? It is fair to say that we could have discovered and simulated the ELB contagion mechanism by using exclusively linearization methods. But we would not know how reliable our linear solution is, and we would miss some dramatic effects of nonlinearities on the predictions of the bToTEM model. In particular, the uncertainty effect makes the steady states of linearized and nonlinear versions of the model to differ, so that the linearized and nonlinear models with the same initial condition lead to very different transitional dynamics. Another important type of nonlinearity in the bToTEM model is a specific form of the closing condition. The existing literature found that this condition plays little role in open-economy models but it plays a dramatic role in the bToTEM dynamics. The effects associated with this condition would be impossible to detect with the linearization analysis since both of the closing conditions we consider have the same linearized form. Thus, having a nonlinear solution is critical. On the other hand, the nonlinearity associated with the ELB turned out to be of a lesser importance than we expected. Central banks pay particular attention to this type of nonlinearity after the Great Recession but our result indicates that this nonlinearity is not necessarily overwhelming. We do not intend to say that the Great Recession was unimportant but that the mechanism that produced that recession in Canada was not dodged by the ELB.

The bToTEM model constructed in the paper provides a useful alternative model to the Bank of Canada. While the full-scale ToTEM is not yet feasible for global nonlinear methods, bToTEM can be solved nonlinearly, and its accuracy can be assessed. But our analysis can be useful to all users of large-scale models, namely, researchers, central banks and government agencies who can benefit from our methodology of calibrating, solving, and simulating large-scale macroeconomic models, as well as for designing nontrivial policy experiments within such models.

References

[1] Adjemian, S., H. Bastani, M. Juillard, F. Mihoubi, G. Perendia, M. Ratto, and S. Villemot, (2011). Dynare: reference manual, Version 4. Dynare Working Papers 1, CEPREMAP.

[2] Azinović, M, L. Gaegauf, and S. Scheidegger. (2019). Deep equilibrium nets. Manuscript. SSRN working paper https://ssrn.com/abstract=3393482

[3] Adolfson, M., S. Laséen, L. Christiano, M. Trabandt, and K. Walentin, (2013). Ramses II – model description. Sveriges Riksbank Occasional Paper Series 12.

[4] Andreasen, M., J. Fernández-Villaverde, and J. Rubio-Ramírez, (2013). The pruned state-space system for non-linear DSGE models: theory and empirical applications. NBER Working Paper 18983.

[5] Aruoba, S., P. Cuba-Borda, and F. Schorfheide, (2018). Macroeconomic dynamics near the ZLB: a tale of two countries. Review of Economic Studies 85 (1), 87–118.

[6] Aruoba, S., J. Fernández-Villaverde, and J. Rubio-Ramírez, (2006). Comparing solution methods for dynamic equilibrium economies. Journal of Economic Dynamics and Control 30, 2477-2508.

[7] Ascari, G., and A. Sbordone, (2014). The macroeconomics of trend inflation. Journal of Economic Literature 52 (3), 678-739.

[8] Backus, D. K., P. J. Kehoe, and F. E. Kydland, (1992). International real business cycles. Journal of Political Economy 100 (4), 745-775.

[9] Ball, L., (2013). The case for four percent inflation. Central Bank Review 13 (2): 17-31.

[10] Bayoumi, T., D. Laxton, and P. Pesenti, (2001). Benefits and spillovers of greater competition in Europe: A macroeconomic assessment. European Central Bank Working Paper No. 341.

[11] Beneš, J., (2015). Equation-selective nonlinear simulations. Manuscript.

[12] Beneš, J., M. Johnston, and S. Plotnikov, (2015). IRIS toolbox release 20151016 (Macroeconomic modeling toolbox), software available at `http://www.iris-toolbox.com`.

[13] Binette, A., S. Murchison, P. Perrier, and A. Rennison, (2004). An introduction to TOTEM. Bank of Canada. Manuscript.

[14] Blanchard, O., G. Dell'Ariccia, and P. Mauro, (2010). Rethinking macroeconomic policy. International Monetary Fund Staff Position Note, SPN/10/03.

[15] Bodenstein, M., C. Erceg, and L. Guerrieri, (2016). The effects of foreign shocks when interest rates are at zero. International Finance Discussion Papers 983r. Board of Governors of the Federal Reserve System.

[16] Boivin, J., (2011). The "Great" Recession in Canada: perception vs. reality. Speech from March 28, 2011.

[17] Boneva, L., R. Braun, and Y. Waki, (2016). Some unpleasant properties of log-linearized solutions when the nominal rate is zero. Journal of Monetary Economics 84, 216-232.

[18] Brayton, F., A. Levin, R. Tryon, and J. Williams, (1997). The evolution of macro models at the Federal Reserve Board. Carnegie-Rochester Conference Series on Public Policy 47, 43-81.

[19] Bullard, J., (2013). Seven faces of "the peril". Federal Reserve Bank of St. Louis Review, 95.

[20] Burgess, S., E. Fernandez-Corugedo, C. Groth, R. Harrison, F. Monti, K. Theodoridis, and M. Waldron, (2013). The Bank of England's forecasting platform: COMPASS, MAPS, EASE and the suite of models. Working Paper 471.

[21] Caballero, J., E. Farhi, and P. Gourinchas, (2016). Global imbalances and currency wars at the ZLB. Manuscript.

[22] Christiano, L., M. Eichenbaum, and B. Johannsen, (2016). Does the new Keynesian model have a uniqueness problem? Manuscript.

[23] Christiano, L., M. Eichenbaum, and M. Trabandt, (2015). Understanding the Great Recession. American Economic Journal: Macroeconomics 7 (1), 110-167.

[24] Chung, H., J. Laforte, D. Reifschneider, and J. C. Williams, (2012). Have we underestimated the likelihood and severity of zero lower bound events? Journal of Money, Credit and Banking 44 (Supplement 1), 47-82.

[25] Coenen, G., P. McAdam, and R. Straub, (2008). Tax reform and labour-market performance in the Euro Area: a simulation-based analysis using the New Area-Wide Model. Journal of Economic Dynamics and Control 32, 2543–2583.

[26] Coleman, C., Lyon, S., Maliar, L. and S. Maliar, (2018). Matlab, python, julia: what to choose in economics? CEPR discussion paper DP 13210.

[27] Coibion, O., Y. Gorodnichenko, and J. Wieland, (2012). The optimal inflation rate in new Keynesian models: should central banks raise their inflation targets in light of the zero lower bound. Review of Economic Studies 79, 1371-1406.

[28] Cook, D., and M. Devereux, (2011). Global vs. local liquidity traps. Seoul Journal of Economics, 24 (4), 471-493.

[29] Cook, D., and M. Devereux, (2013). Sharing the burden: monetary and fiscal responses to a world liquidity trap. American Economic Journal: Macroeconomics, 5 (3), 190-228.

[30] Cook, D. and M. Devereux, (2016). Exchange rate flexibility under the zero lower bound. Journal of International Economics 101, 52-69.

[31] Corsetti, G., K. Kuester, and G. Müller, (2016). The case for flexible exchange rates in a Great Recession. Manuscript.

[32] Côté, A., (2014). Inflation targeting in the post-crisis era. Speech from November 18, 2014.

[33] Debortoli, D., J. Gali and L. Gambetti, (2019). On the empirical (ir)relevance of the zero lower bound constraint. Manuscript.

[34] Devereux, M. (2014). International aspects of the zero lower bound constraint. In: Macroeconomic and Financial Stability: Challenges for Monetary Policy, edited by S. Bauducco, L. Christiano, and C. Raddatz, 383-425.

[35] Dorich, J., M. Johnston, R. Mendes, S. Murchison, and Y. Zhang, (2013). ToTEM II: An updated version of the Bank of Canada's Quarterly Projection Model. Bank of Canada Technical Report No. 100.

[36] Dorich, J., N. Labelle, V. Lepetyuk, and R. Mendes, (2018). Could a higher inflation target enhance macroeconomic stability? Canadian Journal of Economics, 51 (3), 1029-1055.

[37] Duarte, V. (2018). Machine learning for continuous-time economics. Manuscript.

[38] Duffy, J. and P. McNelis, (2001). Approximating and simulating the real business cycle model: parameterized expectations, neural networks, and the genetic algorithm, Journal of Economic Dynamics and Control 25(9), 1273-1303.

[39] Eggertsson, G. B., N. R. Mehrotra, S. R. Singh, and L. Summers, (2016). A contagious malady? Open economy dimensions of secular stagnation. NBER Working Paper 22299.

[40] Eggertsson, G. B. and S. R. Singh, (2016). Log-linear approximation versus an exact solution at the ZLB in the new Keynesian model. NBER Working Paper 22784.

[41] Erceg, C. J., L. Guerrieri, and C. Gust, (2006). SIGMA: A new open economy model for policy analysis. International Journal of Central Banking 2 (1), 1-50.

[42] Fernandez, A., S. Schmitt-Grohé, and M. Uribe, (2017). World shocks, world prices, and business cycles: an empirical investigation. Journal of International Economics 108, Supplement 1, S2-S14.

[43] Fernández-Villaverde, J., G. Gordon, P. Guerrón-Quintana, and J. Rubio-Ramírez, (2015). Nonlinear adventures at the zero lower bound. Journal of Economic Dynamics and Control 57, 182-204.

[44] Fernández-Villaverde, J., S. Hurtado, and G. Nuño, (2018). Financial frictions and the wealth distribution. Manuscript.

[45] Fischer, S., (1996). Why are central banks pursuing long-run price stability? Paper presented at the Achieving Price Stability symposium sponsored by the Federal Reserve Bank of Kansas City, Jackson Hole, August 29-31.

[46] Fujiwara, I., (2010). Export shocks and the zero bound trap. Federal Reserve Bank of Dallas, Working Paper 63.

[47] Galí, J., and Gertler, (1999). Inflation dynamics: A structural econometric analysis. Journal of Monetary Economics 44 (2), 195-222.

[48] Gervais, O., and M. Gosselin, (2014). Analyzing and forecasting the Canadian economy through the LENS model. Bank of Canada Technical Report No. 102.

[49] Goodfellow, I., Y. Bengio, and A. Courville, (2016). Deep Learning. Massachusetts Institute Technology Press.

[50] Guerrieri, L., and M. Iacoviello, (2015). OccBin: A toolkit for solving dynamic models with occasionally binding constraints easily. Journal of Monetary Economics 70, 22-38.

[51] Gust, C., D. Lopez-Salido, and M. Smith, (2012). The empirical implications of the interest-rate lower bound. Federal Reserve Board, Finance and Economics Discussion Series, No. 2012-83.

[52] Hills, T., T. Nakata, and S. Schmidt, (2016). The Risky Steady State and the Interest Rate Lower Bound, Finance and Economics Discussion Series 2016-009, Board of Governors of the Federal Reserve System

[53] Holden, T., (2016). Computation of solutions to dynamic models with occasionally binding constraints. University of Surrey, Manuscript.

[54] Jeanne, O., (2010). The global liquidity trap. Revue Économique 61 (3), 395-406.

[55] Judd, K., L. Maliar, and S. Maliar, (2011a). A cluster-grid projection algorithm: solving problems with high dimensionality. Manuscript. http://www.stanford.edu/~maliarl/Files/MS_9268_R.pdf.

[56] Judd, K., L. Maliar, and S. Maliar, (2011b). Numerically stable and accurate stochastic simulation approaches for solving dynamic models. Quantitative Economics 2 (2), 173-210.

[57] Judd, K., L. Maliar, and S. Maliar, (2012). Merging simulation and projection approaches to solve high-dimensional problems. NBER Working Paper 18501.

[58] Judd, K., L. Maliar, and S. Maliar, (2017). Lower bounds on approximation errors to numerical solutions of dynamic economic models. Econometrica 85 (3), 991-1012.

[59] Kollmann, R., S. Maliar, B. Malin, and P. Pichler, (2011). Comparison of solutions to the multi-country real business cycle model. Journal of Economic Dynamics and Control 35, 186-202.

[60] Krugman, P., (1998). It's baaack: Japan's slump and the return of the liquidity trap. Brookings Papers on Economic Activity 2, 137-187.

[61] Krusell, P., and A. Smith, (1998). Income and wealth heterogeneity in the macroeconomy. Journal of Political Economy 106, 868–896.

[62] Kryvtsov, O., and R. Mendes, (2015). The optimal level of the inflation target: a selective review of the literature and outstanding issues, Bank of Canada Staff Discussion Paper No. 2015-8.

[63] Laséen, S., and L. Svensson, (2011). Anticipated alternative policy rate paths in policy simulations. International Journal of Central Banking 7 (3), 1-35.

[64] Leahy, J., (2013). Like a good neighbor: the importance of non-linearities and expectations in the recent crisis. International Journal of Central Banking 9 (2), 287-293.

[65] Lepetuyk, V., Maliar, L. and S. Maliar (2017). Should central banks worry about nonlinearities of their large-scale macroeconomic models? Bank of Canada staff paper #2017-21.

[66] Lim, G., and P. McNelis, (2008). Computational macroeconomics for the open economy. London, England: The MIT Press, Cambridge, Massachusetts.

[67] Lubik, T., and F. Schorfheide, (2005). A Bayesian look at new open economy macroeconomics. In NBER Macroeconomics Annual 2005, edited by Mark Gertler, and Kenneth Rogoff, vol. 20. Cambridge: MIT Press.

[68] Maliar, L., and S. Maliar, (2014). Numerical methods for large scale dynamic economic models. In Handbook of Computational Economics, edited by K. Schmedders and K. Judd , vol. 3. Elsevier Science, Amsterdam

[69] Maliar, L., and S. Maliar, (2015). Merging simulation and projection approaches to solve high-dimensional problems with an application to a new Keynesian model. Quantitative Economics 6 (1), 1-47.

[70] Maliar, L., S. Maliar, and K. L. Judd, (2011). Solving the multi-country real business cycle model using ergodic set methods. Journal of Economic Dynamic and Control 35 (2), 207-228.

[71] Maliar, L., S. Maliar, and S. Villemot, (2013). Taking perturbation to the accuracy frontier: a hybrid of local and global solutions. Computational Economics 42 (3), 307-325.

[72] Maliar, L., S. Maliar, and P. Winant, (2018). Deep learning for solving dynamic economic models. Manuscript. https://lmaliar.ws.gc.cuny.edu/.

[73] Maliar, L., S. Maliar, and P. Winant, (2019). Will artificial intelligence replace computational economists any time soon? CEPR discussion paper DP14024.

[74] McNelis, P. (2005). Neural Networks in Finance: Gaining Predictive Edge in the Market. Elsevier Academic Press.

[75] Mendoza, E. (1991). Real business cycles in a small-open economy. American Economic Review 81, 797-818.

[76] Mertens, K., and M. Ravn, (2014). Fiscal policy in an expectations driven liquidity trap. Review of Economic Studies 81, 1637-1667.

[77] Murchison, S., and A. Rennison, (2006). ToTEM: The Bank of Canada's new quarterly projection model. Bank of Canada Technical Report No. 97.

[78] Renewal of the Inflation-Control Target. Background Information (2016), Bank of Canada, report, October 2016.

[79] Schmitt-Grohé, S., (1998). The international transmission of economic fluctuations: effects of U.S. Business cycles on the Canadian economy. Journal of International Economics 44, 257-287.

[80] Schmitt-Grohé, S., and M. Uribe, (2003). Closing small open economy models. Journal of International Economics 61, 163-185.

[81] Schmitt-Grohé S., and M. Uribe, (2004). Solving dynamic general equilibrium models using a second-order approximation to the policy function. Journal of Economic Dynamics and Control 28 (4), 755-775.

[82] Schmitt-Grohé, S., and M. Uribe, (2012). The making of a great contraction with a liquidity trap and a jobless recovery, NBER Working Papers 18544.

[83] Summers, L., (1991). Panel discussion: price stability: how should long-term monetary policy be determined? Journal of Money, Credit and Banking 23 (3 Part 2): 625-631.

[84] Villa, A. and V. Valaitis (2019). Machine learning projection methods for macro-finance models, Manuscript

[85] Williams, J., (2009), Heeding Daedalus: optimal inflation and the zero lower bound. Brookings Papers on Economic Activity 2, 1-37.

A Derivation of the optimality conditions

In this appendix, we elaborate the derivation of the optimality conditions.

A.1 Production of final goods

First stage of production The Lagrangian of the problem is

$$E_0 \sum_{t=0}^{\infty} \mathcal{R}_{0,t} \left(P_t^z \left(\left(\delta_l \left(A_t L_t\right)^{\frac{\sigma-1}{\sigma}} + \delta_k \left(u_t K_t\right)^{\frac{\sigma-1}{\sigma}} + \delta_{com} \left(COM_t^d\right)^{\frac{\sigma-1}{\sigma}} + \delta_m \left(M_t\right)^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}} - \frac{\chi_i}{2} \left(\frac{I_t}{I_{t-1}} - 1 \right)^2 I_t \right) - W_t L_t - P_t^{com} COM_t^d - P_t^i I_t - P_t^m M_t + Q_t \left(\left(1 - d_t\right) K_t + I_t - K_{t+1} \right) \right).$$

The optimal quantities satisfy the following conditions, with an augmented discount factor as in ToTEM:

$$W_t = P_t^z \left(Z_t^g\right)^{\frac{1}{\sigma}} \delta_l \left(A_t\right)^{\frac{\sigma-1}{\sigma}} \left(L_t\right)^{\frac{-1}{\sigma}} \tag{A.1}$$

$$Q_t = \frac{1}{R_t^k} E_t \left[P_{t+1}^z \left(Z_{t+1}^g\right)^{\frac{1}{\sigma}} \delta_k \left(u_{t+1}\right)^{\frac{\sigma-1}{\sigma}} \left(K_{t+1}\right)^{\frac{-1}{\sigma}} + Q_{t+1} \left(1 - d_{t+1}\right) \right] \tag{A.2}$$

$$P_t^i = Q_t - P_t^z \frac{\chi_i}{2} \left(\frac{I_t}{I_{t-1}} - 1 \right) \left(\frac{3 I_t}{I_{t-1}} - 1 \right) + \frac{1}{R_t^k} E_t \left[P_{t+1}^z \chi_i \left(\frac{I_{t+1}}{I_t} - 1 \right) \left(\frac{I_{t+1}}{I_t} \right)^2 \right] \tag{A.3}$$

$$P_t^{com} = P_t^z \left(Z_t^g\right)^{\frac{1}{\sigma}} \delta_{com} \left(COM_t^d\right)^{\frac{-1}{\sigma}} \tag{A.4}$$

$$P_t^m = P_t^z \left(Z_t^g\right)^{\frac{1}{\sigma}} \delta_m \left(M_t\right)^{\frac{-1}{\sigma}} \tag{A.5}$$

$$Q_t \bar{d} \rho e^{\rho(u_t - 1)} = P_t^z \left(Z_t^g\right)^{\frac{1}{\sigma}} \delta_k \left(u_t\right)^{\frac{-1}{\sigma}} \left(K_t\right)^{\frac{-1}{\sigma}} \tag{A.6}$$

where Q_t is the Lagrange multiplier on the law of motion of capital (3). Introducing real prices by $p_t^z = P_t^z / P_t$, $w_t = W_t / P_t$, $p_t^i = P_t^i / P_t$, $p_t^{com} = P_t^{com} / P_t$, $p_t^m = P_t^m / P_t$, and $q_t = Q_t / P_t$, where P_t is the price of final good, the conditions can be written as

$$w_t = p_t^z \left(Z_t^g\right)^{\frac{1}{\sigma}} \delta_l \left(A_t\right)^{\frac{\sigma-1}{\sigma}} \left(L_t\right)^{\frac{-1}{\sigma}} \tag{A.7}$$

$$q_t = \frac{1}{R_t^k} E_t \left[\pi_{t+1} \left(p_{t+1}^z \left(Z_{t+1}^g\right)^{\frac{1}{\sigma}} \delta_k \left(u_{t+1}\right)^{\frac{\sigma-1}{\sigma}} \left(K_{t+1}\right)^{\frac{-1}{\sigma}} + q_{t+1} \left(1 - d_{t+1}\right) \right) \right] \tag{A.8}$$

$$p_t^i = q_t - p_t^z \frac{\chi_i}{2} \left(\frac{I_t}{I_{t-1}} - 1 \right) \left(\frac{3 I_t}{I_{t-1}} - 1 \right) + \frac{1}{R_t^k} E_t \left[\pi_{t+1} p_{t+1}^z \chi_i \left(\frac{I_{t+1}}{I_t} - 1 \right) \left(\frac{I_{t+1}}{I_t} \right)^2 \right] \tag{A.9}$$

$$p_t^{com} = p_t^z \left(Z_t^g\right)^{\frac{1}{\sigma}} \delta_{com} \left(COM_t^d\right)^{\frac{-1}{\sigma}} \tag{A.10}$$

$$p_t^m = p_t^z \left(Z_t^g\right)^{\frac{1}{\sigma}} \delta_m \left(M_t\right)^{\frac{-1}{\sigma}} \tag{A.11}$$

$$q_t \bar{d} \rho e^{\rho(u_t - 1)} = p_t^z \left(Z_t^g\right)^{\frac{1}{\sigma}} \delta_k \left(u_t\right)^{\frac{-1}{\sigma}} \left(K_t\right)^{\frac{-1}{\sigma}} \tag{A.12}$$

Second stage of production The first-order condition associated with the problem (9)-(10) is

$$E_t \left[\sum_{j=0}^{\infty} \theta^j \mathcal{R}_{t,t+j} Z_{i,t+j} \left(\prod_{k=1}^{j} \bar{\pi}_{t+k} (1-\varepsilon) + \varepsilon \frac{(1-s_m) P^z_{t+j} + s_m P_{t+j}}{P^*_t} \right) \right] = 0,$$

or using the demand function (10)

$$E_t \left[\sum_{j=0}^{\infty} (\beta\theta)^j \lambda_{t+j} \left(\frac{\prod_{k=1}^{j} \bar{\pi}_{t+k}}{\prod_{k=1}^{j} \pi_{t+k}} \right)^{-\varepsilon} Z_{t+j} \left(\frac{\prod_{k=1}^{j} \bar{\pi}_{t+k}}{\prod_{k=1}^{j} \pi_{t+k}} \frac{P^*_t}{P_t} - \frac{\varepsilon}{\varepsilon-1} rmc_{t+j} \right) \right] = 0, \tag{A.13}$$

where the real marginal cost is

$$rmc_t = (1-s_m) \frac{P^z_t}{P_t} + s_m. \tag{A.14}$$

The condition (A.13) can be written as

$$\frac{P^*_t}{P_t} = \frac{F_{1t}}{F_{2t}} \tag{A.15}$$

$$F_{1t} \equiv E_t \left[\sum_{j=0}^{\infty} (\beta\theta)^j \lambda_{t+j} \left(\frac{\prod_{k=1}^{j} \bar{\pi}_{t+k}}{\prod_{k=1}^{j} \pi_{t+k}} \right)^{-\varepsilon} Z_{t+j} \frac{\varepsilon}{\varepsilon-1} rmc_{t+j} \right]$$

$$F_{2t} \equiv E_t \left[\sum_{j=0}^{\infty} (\beta\theta)^j \lambda_{t+j} \left(\frac{\prod_{k=1}^{j} \bar{\pi}_{t+k}}{\prod_{k=1}^{j} \pi_{t+k}} \right)^{1-\varepsilon} Z_{t+j} \right].$$

The equations for F_{1t} and F_{2t} can be written recursively as

$$F_{1t} = \lambda_t Z_t \frac{\varepsilon}{\varepsilon-1} rmc_t + \beta\theta E_t \left[\left(\frac{\bar{\pi}_{t+1}}{\pi_{t+1}} \right)^{-\varepsilon} F_{1t+1} \right], \tag{A.16}$$

$$F_{2t} = \lambda_t Z_t + \beta\theta E_t \left[\left(\frac{\bar{\pi}_{t+1}}{\pi_{t+1}} \right)^{1-\varepsilon} F_{2t+1} \right]. \tag{A.17}$$

The aggregate price introduced by (7) satisfies the following condition:

$$\theta \left(\frac{\bar{\pi}_t}{\pi_t} \right)^{1-\varepsilon} + (1-\theta)\,\omega \left(\frac{(\pi_{t-1})^{\gamma} (\bar{\pi}_t)^{1-\gamma}}{\pi_t} \right)^{1-\varepsilon} + (1-\theta)(1-\omega) \left(\frac{P^*_t}{P_t} \right)^{1-\varepsilon} = 1. \tag{A.18}$$

Combining (A.18) with the optimal price setting (A.15), we get

$$\theta \left(\frac{\bar{\pi}_t}{\pi_t} \right)^{1-\varepsilon} + (1-\theta)\,\omega \left(\frac{(\pi_{t-1})^{\gamma} (\bar{\pi}_t)^{1-\gamma}}{\pi_t} \right)^{1-\varepsilon} + (1-\theta)(1-\omega) \left(\frac{F_{1t}}{F_{2t}} \right)^{1-\varepsilon} = 1. \tag{A.19}$$

Relation between the first and second stages of production Introducing the following price index

$$\bar{P}_t = \left(\int_0^1 P_{it}^{-\varepsilon} di \right)^{\frac{-1}{\varepsilon}},$$

which can be expressed using the price settings of the rule-of-thumb and forward-looking firms as follows:

$$\left(\frac{\bar{P}_t}{P_t} \right)^{-\varepsilon} = \theta \left(\frac{\bar{\pi} \bar{P}_{t-1}}{P_t} \right)^{-\varepsilon} + (1-\theta)\,\omega \left(\frac{(\pi_{t-1})^{\gamma} (\bar{\pi}_t)^{1-\gamma} \bar{P}_{t-1}}{P_t} \right)^{-\varepsilon} + (1-\theta)(1-\omega) \left(\frac{P^*_t}{P_t} \right)^{-\varepsilon},$$

the dynamics of the price dispersion is the following:

$$\Delta_t = \theta \left(\frac{\bar{\pi}_t}{\pi_t} \right)^{-\varepsilon} \Delta_{t-1} + (1-\theta)\,\omega \left(\frac{(\pi_{t-1})^{\gamma} (\bar{\pi}_t)^{1-\gamma}}{\pi_t} \right)^{-\varepsilon} \Delta_{t-1} + (1-\theta)(1-\omega) \left(\frac{F_{1t}}{F_{2t}} \right)^{-\varepsilon}. \tag{A.20}$$

A.2 Commodities

The Lagrangian of the problem is the following:

$$E_0 \sum_{t=0}^{\infty} \mathcal{R}_{0,t} \left(P_t^{com} \left(\left(Z_t^{com}\right)^{s_z} \left(A_t L\right)^{1-s_z} - \frac{\chi_{com}}{2} \left(\frac{Z_t^{com}}{Z_{t-1}^{com}} - 1 \right)^2 Z_t^{com} \right) - P_t Z_t^{com} \right).$$

The resulting partial adjustment equation for the commodity-producing firm is as follows:

$$P_t = P_t^{com} \frac{s_z COM_t}{Z_t^{com}} - P_t^{com} \frac{\chi_{com}}{2} \left(\frac{Z_t^{com}}{Z_{t-1}^{com}} - 1 \right) \left(\frac{3Z_t^{com}}{Z_{t-1}^{com}} - 1 \right) + \frac{1}{R_t} E_t \left[P_{t+1}^{com} \chi_{com} \left(\frac{Z_{t+1}^{com}}{Z_t^{com}} - 1 \right) \left(\frac{Z_{t+1}^{com}}{Z_t^{com}} \right)^2 \right], \tag{A.21}$$

or expressed in real prices

$$1 = p_t^{com} \frac{s_z COM_t}{Z_t^{com}} - p_t^{com} \frac{\chi_{com}}{2} \left(\frac{Z_t^{com}}{Z_{t-1}^{com}} - 1 \right) \left(\frac{3Z_t^{com}}{Z_{t-1}^{com}} - 1 \right) + \frac{1}{R_t} E_t \left[\pi_{t+1} p_{t+1}^{com} \chi_{com} \left(\frac{Z_{t+1}^{com}}{Z_t^{com}} - 1 \right) \left(\frac{Z_{t+1}^{com}}{Z_t^{com}} \right)^2 \right]. \tag{A.22}$$

A.3 Imports

Similarly to (A.13), the first-order optimality condition associated with the problem of optimizing forward-looking importers is the following:

$$E_t \left[\sum_{j=0}^{\infty} \left(\beta \theta_m\right)^j \lambda_{t+j} M_{i,t+j} \left(\frac{\prod_{k=1}^{j} \bar{\pi}_{t+k}}{\prod_{k=1}^{j} \pi_{t+k}} \frac{P_t^{m*}}{P_t} - \frac{\varepsilon_m}{\varepsilon_m - 1} \frac{e_{t+j} P_{t+j}^{mf}}{P_{t+j}} \right) \right] = 0. \tag{A.23}$$

The condition (A.23) can be written as

$$\frac{P_t^{m*}}{P_t} = \frac{F_{1t}^m}{F_{2t}^m}, \tag{A.24}$$

where F_{1t}^m and F_{2t}^m are given by the following equations:

$$F_{1t}^m = \lambda_t M_t \frac{\varepsilon_m}{\varepsilon_m - 1} s_t p_t^{mf} + \beta \theta_m E_t \left[\left(\frac{\bar{\pi}_{t+1}}{\pi_{t+1}^m} \right)^{-\varepsilon_m} F_{1t+1}^m \right], \tag{A.25}$$

$$F_{2t}^m = \lambda_t M_t + \beta \theta_m E_t \left[\frac{\bar{\pi}_{t+1}}{\pi_{t+1}} \left(\frac{\bar{\pi}_{t+1}}{\pi_{t+1}^m} \right)^{-\varepsilon_m} F_{2t+1}^m \right], \tag{A.26}$$

and where s_t and p_t^{mf} are the real exchange rate and the real foreign price of imports introduced by $s_t = e_t P_t^f / P_t$ and $p_t^{mf} = P_t^{mf} / P_t^f$, respectively. The aggregate import price satisfies the following condition:

$$\theta_m \left(\frac{\bar{\pi}_t}{\pi_t^m} \right)^{1-\varepsilon_m} + \left(1 - \theta_m\right) \omega_m \left(\frac{\left(\pi_{t-1}^m\right)^{\gamma_m} \left(\bar{\pi}_t\right)^{1-\gamma_m}}{\pi_t^m} \right)^{1-\varepsilon_m} + \left(1 - \theta_m\right) \left(1 - \omega_m\right) \left(\frac{P_t^{m*}}{P_t^m} \right)^{1-\varepsilon_m} = 1. \tag{A.27}$$

Combining (A.27) with (A.24), we get

$$\theta_m \left(\frac{\bar{\pi}_t}{\pi_t^m} \right)^{1-\varepsilon_m} + \left(1 - \theta_m\right) \omega_m \left(\frac{\left(\pi_{t-1}^m\right)^{\gamma_m} \left(\bar{\pi}_t\right)^{1-\gamma_m}}{\pi_t^m} \right)^{1-\varepsilon_m} + \left(1 - \theta_m\right) \left(1 - \omega_m\right) \left(\frac{F_{1t}^m}{p_t^m F_{2t}^m} \right)^{1-\varepsilon_m} = 1. \tag{A.28}$$

A.4 Households

The maximization of the lifetime utility (16) subject to the budget constraint (17) with respect to consumption and bond holdings yields the following first-order condition:

$$E_t\left[\beta\frac{\lambda_{t+1}}{\lambda_t}\frac{R_tP_t}{P_{t+1}}\right]=1, \tag{A.29}$$

where λ_t is the marginal utility of consumption, which is given by

$$\lambda_t=\left(C_t-\xi C_{t-1}\right)^{\frac{-1}{\mu}}\exp\left(\frac{\eta\left(1-\mu\right)}{\mu\left(1+\eta\right)}\int_0^1\left(L_{ht}\right)^{\frac{\eta+1}{\eta}}dh\right)\eta_t^c. \tag{A.30}$$

The no-arbitrage condition on holdings of domestic and foreign bonds would imply the following interest rate parity:

$$e_t=E_t\left[\frac{e_{t+1}R_t^f\left(1+\kappa_t^f\right)}{R_t}\right],$$

which is further augmented as in ToTEM to improve business cycle properties of the model as follows:

$$e_t=E_t\left[\left(e_{t-1}\right)^{\varkappa}\left(e_{t+1}\frac{R_t^f\left(1+\kappa_t^f\right)}{R_t}\right)^{1-\varkappa}\right]. \tag{A.31}$$

The condition can be expressed in terms of real exchange rate $s_t=e_tP_t^f/P_t$ as follows:

$$s_t=E_t\left[\left(s_{t-1}\frac{\pi_t^f}{\pi_t}\right)^{\varkappa}\left(s_{t+1}\frac{R_t^f\left(1+\kappa_t^f\right)}{R_t}\frac{\pi_{t+1}}{\pi_{t+1}^f}\right)^{1-\varkappa}\right]. \tag{A.32}$$

A.5 Wage setting

The first-order optimality condition associated with the problem (21)-(22) is

$$E_t\left[\sum_{j=0}^{\infty}\left(\beta\theta_w\right)^j\left(U_{C,t+j}\frac{\prod_{k=1}^j\bar{\pi}_{t+k}}{P_{t+j}}\left(L_{h,t+j}+W_t^*\frac{\partial L_{h,t+j}}{\partial W_t^*}\right)+U_{Lh,t+j}\frac{\partial L_{h,t+j}}{\partial W_t^*}\right)\right]=0,$$

where

$$U_{Ct}=\left(C_t-\xi C_{t-1}\right)^{\frac{-1}{\mu}}\exp\left(\frac{\eta\left(1-\mu\right)}{\mu\left(1+\eta\right)}\int_0^1\left(L_{ht}\right)^{\frac{\eta+1}{\eta}}dh\right)\eta_t^c$$

$$U_{Lh,t}=-\left(C_t-\xi C_{t-1}\right)^{\frac{\mu-1}{\mu}}\exp\left(\frac{\eta\left(1-\mu\right)}{\mu\left(1+\eta\right)}\int_0^1\left(L_{ht}\right)^{\frac{\eta+1}{\eta}}dh\right)\left(L_{ht}\right)^{\frac{1}{\eta}}\eta_t^c$$

$$\frac{\partial L_{h,t+j}}{\partial W_t^*}=\frac{-\varepsilon_w}{W_t^*}L_{h,t+j}$$

or

$$E_t\left[\sum_{j=0}^{\infty}\left(\beta\theta_w\right)^jU_{C,t+j}L_{h,t+j}\left(\frac{\prod_{k=1}^j\bar{\pi}_{t+k}}{\prod_{k=1}^j\pi_{t+k}}\frac{W_t^*}{P_t}-\frac{\varepsilon_w}{\varepsilon_w-1}MRS_{h,t+j}\right)\right]=0, \tag{A.33}$$

where the marginal rate of substitution between consumption and labor is introduced as follows:

$$MRS_{h,t}\equiv\frac{-U_{Lh,t}}{U_{C,t}}=\left(C_t-\xi C_{t-1}\right)\left(L_{ht}\right)^{\frac{1}{\eta}}.$$

Using the demand (22), we write the optimality condition (A.33) as

$$E_t \left[\sum_{j=0}^{\infty} (\beta\theta_w)^j \lambda_{t+j} \left(\frac{\prod_{k=1}^{j} \bar{\pi}_{t+k}}{\prod_{k=1}^{j} \pi_{t+k}} \left(\frac{\prod_{k=1}^{j} \bar{\pi}_{t+k} W_t^*}{\prod_{k=1}^{j} \pi_{t+k}^w W_t} \right)^{-\varepsilon_w} L_{t+j} \frac{W_t^*}{P_t} \right. \right.$$
$$\left. \left. - \frac{\varepsilon_w}{\varepsilon_w - 1} \left(C_{t+j} - \xi C_{t+j-1} \right) \left(\frac{\prod_{k=1}^{j} \bar{\pi}_{t+k} W_t^*}{\prod_{k=1}^{j} \pi_{t+k}^w W_t} \right)^{\frac{-\varepsilon_w(1+\eta)}{\eta}} \left(L_{t+j} \right)^{\frac{1+\eta}{\eta}} \right) \right] = 0,$$

or

$$E_t \left[\sum_{j=0}^{\infty} (\beta\theta_w)^j \lambda_{t+j} \left(\frac{\prod_{k=1}^{j} \bar{\pi}_{t+k}}{\prod_{k=1}^{j} \pi_{t+k}} \left(\frac{\prod_{k=1}^{j} \bar{\pi}_{t+k}}{\prod_{k=1}^{j} \pi_{t+k}^w} \right)^{-\varepsilon_w} L_{t+j} \left(\frac{W_t^*}{W_t} \right)^{\frac{\varepsilon_w}{\eta}} \frac{W_t^*}{P_t} \right. \right.$$
$$\left. \left. - \frac{\varepsilon_w}{\varepsilon_w - 1} \left(\frac{\prod_{k=1}^{j} \bar{\pi}_{t+k}}{\prod_{k=1}^{j} \pi_{t+k}^w} \right)^{\frac{-\varepsilon_w(1+\eta)}{\eta}} \left(C_{t+j} - \xi C_{t+j-1} \right) \left(L_{t+j} \right)^{\frac{1+\eta}{\eta}} \right) \right] = 0. \quad \text{(A.34)}$$

Introducing $w_t^* = W_t^*/P_t$, the condition (A.34) can be stated as

$$(w_t^*)^{1+\frac{\varepsilon_w}{\eta}} (w_t)^{\frac{-\varepsilon_w}{\eta}} = \frac{F_{1t}^w}{F_{2t}^w} \quad \text{(A.35)}$$

$$F_{1t}^w \equiv E_t \left[\sum_{j=0}^{\infty} (\beta\theta_w)^j \lambda_{t+j} \frac{\varepsilon_w}{\varepsilon_w - 1} \left(\frac{\prod_{k=1}^{j} \bar{\pi}_{t+k}}{\prod_{k=1}^{j} \pi_{t+k}^w} \right)^{\frac{-\varepsilon_w(1+\eta)}{\eta}} \left(C_{t+j} - \xi C_{t+j-1} \right) \left(L_{t+j} \right)^{\frac{1+\eta}{\eta}} \right]$$

$$F_{2t}^w \equiv E_t \left[\sum_{j=0}^{\infty} (\beta\theta_w)^j \lambda_{t+j} \frac{\prod_{k=1}^{j} \bar{\pi}_{t+k}}{\prod_{k=1}^{j} \pi_{t+k}} \left(\frac{\prod_{k=1}^{j} \bar{\pi}_{t+k}}{\prod_{k=1}^{j} \pi_{t+k}^w} \right)^{-\varepsilon_w} L_{t+j} \right]$$

The equations for F_{1t}^w and F_{2t}^w can be written recursively as

$$F_{1t}^w = \lambda_t \frac{\varepsilon_w}{\varepsilon_w - 1} \left(C_t - \xi C_{t-1} \right) \left(L_t \right)^{\frac{1+\eta}{\eta}} + \beta\theta_w E_t \left[\left(\frac{\bar{\pi}_{t+1}}{\pi_{t+1}^w} \right)^{\frac{-\varepsilon_w(1+\eta)}{\eta}} F_{1,t+1}^w \right] \quad \text{(A.36)}$$

and

$$F_{2t}^w = \lambda_t L_t + \beta\theta_w E_t \left[\frac{\bar{\pi}_{t+1}}{\pi_{t+1}} \left(\frac{\bar{\pi}_{t+1}}{\pi_{t+1}^w} \right)^{-\varepsilon_w} F_{2,t+1}^w \right]. \quad \text{(A.37)}$$

The aggregate wage defined by (19) satisfies the following condition:

$$\theta_w \left(\frac{\bar{\pi}_t}{\pi_t^w} \right)^{1-\varepsilon_w} + (1-\theta_w)\,\omega_w \left(\frac{\left(\pi_{t-1}^w\right)^{\gamma_w} \left(\bar{\pi}_t\right)^{1-\gamma_w}}{\pi_t^w} \right)^{1-\varepsilon_w} + (1-\theta_w)(1-\omega_w) \left(\frac{W_t^*}{W_t} \right)^{1-\varepsilon_w} = 1. \quad \text{(A.38)}$$

Combining (A.38) with price settings of the optimizing labor unions (A.35), we get

$$\theta_w \left(\frac{\bar{\pi}_t}{\pi_t^w} \right)^{1-\varepsilon_w} + (1-\theta_w)\,\omega_w \left(\frac{\left(\pi_{t-1}^w\right)^{\gamma_w} \left(\bar{\pi}_t\right)^{1-\gamma_w}}{\pi_t^w} \right)^{1-\varepsilon_w} + (1-\theta_w)(1-\omega_w) \left(\frac{w_t^*}{w_t} \right)^{1-\varepsilon_w} = 1. \quad \text{(A.39)}$$

The marginal utility of consumption (A.30) can be expressed employing (18) as follows:

$$\lambda_t = \left(C_t - \xi C_{t-1} \right)^{\frac{-1}{\mu}} \exp\left(\frac{\eta(1-\mu)}{\mu(1+\eta)} \int_0^1 \left(L_{ht} \right)^{\frac{\eta+1}{\eta}} dh \right) \eta_t^c =$$
$$\left(C_t - \xi C_{t-1} \right)^{\frac{-1}{\mu}} \exp\left(\frac{\eta(1-\mu)}{\mu(1+\eta)} \Delta_t^w L_t^{\frac{\eta+1}{\eta}} \right) \eta_t^c, \quad \text{(A.40)}$$

where $\Delta_t^w = \int_0^1 \left(\frac{W_{ht}}{W_t}\right)^{\frac{-\varepsilon_w(\eta+1)}{\eta}} dh$ is the wage dispersion term. Introducing the following price index

$$\bar{W}_t = \left(\int_0^1 (W_{ht})^{\frac{-\varepsilon_w(\eta+1)}{\eta}} dh \right)^{\frac{-\eta}{\varepsilon_w(\eta+1)}},$$

the dynamics of the wage dispersion term can be derived as follows:

$$\begin{aligned} \left(\frac{\bar{W}_t}{W_t}\right)^{\frac{-\varepsilon_w(\eta+1)}{\eta}} &= \theta_w \left(\frac{\bar{\pi}\bar{W}_{t-1}}{W_t}\right)^{\frac{-\varepsilon_w(\eta+1)}{\eta}} \\ &+ (1-\theta_w)\,\omega_w \left(\frac{\left(\pi_{t-1}^w\right)^{\gamma_w} (\bar{\pi}_t)^{1-\gamma_w}\, \bar{W}_{t-1}}{W_t}\right)^{\frac{-\varepsilon_w(\eta+1)}{\eta}} + (1-\theta_w)(1-\omega_w)\left(\frac{W_t^*}{W_t}\right)^{\frac{-\varepsilon_w(\eta+1)}{\eta}} \end{aligned}$$

or

$$\begin{aligned} \Delta_t^w = \theta_w &\left(\frac{\bar{\pi}_t}{\pi_t^w}\right)^{\frac{-\varepsilon_w(\eta+1)}{\eta}} \Delta_{t-1}^w \\ &+ (1-\theta_w)\,\omega_w \left(\frac{\left(\pi_{t-1}^w\right)^{\gamma_w} (\bar{\pi}_t)^{1-\gamma_w}}{\pi_t^w}\right)^{\frac{-\varepsilon_w(\eta+1)}{\eta}} \Delta_{t-1}^w + (1-\theta_w)(1-\omega_w)\left(\frac{w_t^*}{w_t}\right)^{\frac{-\varepsilon_w(\eta+1)}{\eta}}. \quad \text{(A.41)} \end{aligned}$$

B List of model variables

In Table B.1, we list 49 endogenous model variables. When solving the models, the variables are taken either in levels or in logarithms as stated in the table.

Variable	Symbol	In logarithms
productivity	A_t	no
labor input	L_t	yes
capital input	K_t	yes
investment	I_t	yes
commodities used domestically	COM_t^d	yes
import	M_t	yes
capital utilization	u_t	no
capital depreciation	d_t	no
gross production of intermediate good	Z_t^g	yes
net production of intermediate good	Z_t^n	yes
total production	Z_t	yes
consumption	C_t	yes
GDP	Y_t	yes
marginal utility of consumption	λ_t	yes
inflation	π_t	yes
real marginal cost	rmc_t	yes
consumption Phillips curve term	F_{1t}	yes
consumption Phillips curve term	F_{1t}	yes
price dispersion	Δ_t	yes
imported good inflation	π_t^m	yes
imports Phillips curve term	F_{1t}^m	yes
imports Phillips curve term	F_{1t}^m	yes
inflation target	$\bar{\pi}_t$	yes
real price of import	p_t^m	yes
real exchange rate	s_t	yes
nominal interest rate	R_t	no
interest rate shock process	η_t^r	no
real price of intermediate good	p_t^z	yes
real wage	w_t	yes
real price of commodities	p_t^{com}	yes
marginal product of capital	MPK_t	yes
interest rate on capital	R_t^k	no
Tobin's Q	q_t	yes
real price of investment	p_t^i	yes
foreign-currency price of commodities	p_t^{comf}	no
foreign real interest rate	r_t^f	no
interest premium on foreign bonds	κ_t^f	no
holdings of foreign bonds in real terms	b_t^f	no
non-commodity export	X_t^{nc}	yes
export of commodities	X_t^{com}	yes
foreign demand	Z_t^f	no
total commodities produced	COM_t	yes
final goods used in commodity production	Z_t^{com}	yes
wage inflation	π_t^w	yes

wage Phillips curve term	F^w_{1t}	yes
wage Phillips curve term	F^w_{2t}	yes
optimal wage	w^*_t	yes
wage dispersion	Δ^w_t	yes
auxiliary expectation term	ex^i_t	no
auxiliary expectation term	ex^{com}_t	no
foreign price of import	p^{mf}_t	yes
GDP deflator	p^y_t	yes
price of non-commodity export	p^{xz}_t	yes
consumption demand shock process	η^c_t	no
potential GDP	$\bar{Y}_t$	yes

Table B.1: A list of model's variables

C List of model equations

The bToTEM model consists of 49 equations and 49 endogenous variables, as well as 6 exogenous autocorrelative shock processes. Here we summarize all model equations.

- Production of finished goods
  - Production technology (1), (5)

$$Z_t^g = \left(\delta_l \left(A_t L_t\right)^{\frac{\sigma-1}{\sigma}} + \delta_k \left(u_t K_t\right)^{\frac{\sigma-1}{\sigma}} + \delta_{com} \left(COM_t^d\right)^{\frac{\sigma-1}{\sigma}} + \delta_m \left(M_t\right)^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}}$$

$$Z_t^n = Z_t^g - \frac{\chi_i}{2}\left(\frac{I_t}{I_{t-1}} - 1\right)^2 I_t$$

  - Optimality conditions (A.1), (A.2), (A.3), (A.4), (A.5), (A.6)

$$w_t = p_t^z \left(Z_t^g\right)^{\frac{1}{\sigma}} \delta_l \left(A_t\right)^{\frac{\sigma-1}{\sigma}} \left(L_t\right)^{\frac{-1}{\sigma}}$$

$$MPK_t = \left(Z_t^g\right)^{\frac{1}{\sigma}} \delta_k \left(u_t\right)^{\frac{-1}{\sigma}} \left(K_t\right)^{\frac{-1}{\sigma}}$$

$$R_t^k = R_t \left(1 + \kappa_t^k\right)$$

$$q_t = \frac{1}{R_t^k} E_t \left[\pi_{t+1} \left(p_{t+1}^z MPK_{t+1} u_{t+1} + q_{t+1} \left(1 - d_{t+1}\right)\right)\right]$$

$$p_t^i = q_t - p_t^z \frac{\chi_i}{2}\left(\frac{I_t}{I_{t-1}} - 1\right)\left(\frac{3I_t}{I_{t-1}} - 1\right) + \frac{1}{R_t^k} E_t \left[\pi_{t+1} p_{t+1}^z \chi_i \left(\frac{I_{t+1}}{I_t} - 1\right)\left(\frac{I_{t+1}}{I_t}\right)^2\right]$$

$$p_t^{com} = p_t^z \left(Z_t^g\right)^{\frac{1}{\sigma}} \delta_{com} \left(COM_t^d\right)^{\frac{-1}{\sigma}}$$

$$p_t^m = p_t^z \left(Z_t^g\right)^{\frac{1}{\sigma}} \delta_m \left(M_t\right)^{\frac{-1}{\sigma}}$$

$$q_t \overline{d} \rho e^{\rho(u_t - 1)} = p_t^z MPK_t$$

  - Law of motion for capital (3), (4)

$$K_t = \left(1 - d_{t-1}\right) K_{t-1} + I_{t-1}$$

$$d_t = d_0 + \overline{d} e^{\rho(u_t - 1)}$$

  - New-Keynesian Phillips curve for finished goods (A.14), (A.16), (A.17), (A.19)

$$rmc_t = p_t^z (1 - s_m) + s_m$$

$$F_{1t} = \lambda_t Z_t \frac{\varepsilon}{\varepsilon - 1} rmc_t + \beta\theta E_t \left[\left(\frac{\bar{\pi}_{t+1}}{\pi_{t+1}}\right)^{-\varepsilon} F_{1t+1}\right]$$

$$F_{2t} = \lambda_t Z_t + \beta\theta E_t \left[\left(\frac{\bar{\pi}_{t+1}}{\pi_{t+1}}\right)^{1-\varepsilon} F_{2t+1}\right]$$

$$\theta \left(\frac{\bar{\pi}_t}{\pi_t}\right)^{1-\varepsilon} + (1-\theta)\,\omega \left(\frac{\left(\pi_{t-1}\right)^{\gamma} \left(\bar{\pi}_t\right)^{1-\gamma}}{\pi_t}\right)^{1-\varepsilon} + (1-\theta)(1-\omega)\left(\frac{F_{1t}}{F_{2t}}\right)^{1-\varepsilon} = 1$$

– Price dispersion (11), (A.20)

$$Z_t^n = (1 - s_m)\,\Delta_t Z_t$$

$$\Delta_t = \theta \left(\frac{\bar{\pi}_t}{\pi_t}\right)^{-\varepsilon} \Delta_{t-1} + (1-\theta)\,\omega \left(\frac{(\pi_{t-1})^{\gamma}\,(\bar{\pi}_t)^{1-\gamma}}{\pi_t}\right)^{-\varepsilon} \Delta_{t-1} + (1-\theta)\,(1-\omega)\left(\frac{F_{1t}}{F_{2t}}\right)^{-\varepsilon}$$

- Commodities

– Production technology (12)

$$COM_t = \left(Z_t^{com}\right)^{s_z} (A_t F)^{1-s_z} - \frac{\chi_{com}}{2}\left(\frac{Z_t^{com}}{Z_{t-1}^{com}} - 1\right)^2 Z_t^{com}$$

– Demand (A.22)

$$1 = p_t^{com}\frac{s_z COM_t}{Z_t^{com}} - p_t^{com}\frac{\chi_{com}}{2}\left(\frac{Z_t^{com}}{Z_{t-1}^{com}} - 1\right)\left(\frac{3Z_t^{com}}{Z_{t-1}^{com}} - 1\right) + \frac{1}{R_t}E_t\left[\pi_{t+1}p_{t+1}^{com}\chi_{com}\left(\frac{Z_{t+1}^{com}}{Z_t^{com}} - 1\right)\left(\frac{Z_{t+1}^{com}}{Z_t^{com}}\right)^2\right]$$

– Commodity price (13)

$$p_t^{com} = s_t p_t^{comf}$$

- Households

– Euler equation (A.29), (A.40)

$$\lambda_t = (C_t - \xi C_{t-1})^{\frac{-1}{\mu}} \exp\left(\frac{\eta\,(1-\mu)}{\mu\,(1+\eta)}\Delta_t^w (L_t)^{\frac{\eta+1}{\eta}}\right)\eta_t^c$$

$$\lambda_t = E_t\left[\lambda_{t+1}\frac{\beta R_t}{\pi_{t+1}}\right]$$

– Wage dispersion (A.41)

$$\Delta_t^w = \theta_w\left(\frac{\bar{\pi}_t}{\pi_t^w}\right)^{\frac{-\varepsilon_w(\eta+1)}{\eta}}\Delta_{t-1}^w + (1-\theta_w)\,\omega_w\left(\frac{\left(\pi_{t-1}^w\right)^{\gamma_w}(\bar{\pi}_t)^{1-\gamma_w}}{\pi_t^w}\right)^{\frac{-\varepsilon_w(\eta+1)}{\eta}}\Delta_{t-1}^w + (1-\theta_w)\,(1-\omega_w)\left(\frac{w_t^*}{w_t}\right)^{\frac{-\varepsilon_w(\eta+1)}{\eta}}$$

– Phillips curve for wage (A.39), (A.35) (A.36), (A.37)

$$\pi_t^w = \frac{w_t}{w_{t-1}}\pi_t$$

$$(w_t^*)^{1+\frac{\varepsilon_w}{\eta}}(w_t)^{\frac{-\varepsilon_w}{\eta}} = \frac{F_{1t}^w}{F_{2t}^w}$$

$$F_{1t}^w = \lambda_t\frac{\varepsilon_w}{\varepsilon_w - 1}(C_t - \xi C_{t-1})(L_t)^{\frac{1+\eta}{\eta}} + \beta\theta E_t\left[\left(\frac{\bar{\pi}_{t+1}}{\pi_{t+1}^w}\right)^{\frac{-\varepsilon_w(1+\eta)}{\eta}} F_{1,t+1}^w\right]$$

$$F_{2t}^{w} = \lambda_t L_t + \beta\theta E_t \left[\frac{\bar{\pi}_{t+1}}{\pi_{t+1}} \left(\frac{\bar{\pi}_{t+1}}{\pi_{t+1}^{w}} \right)^{-\varepsilon_w} F_{2,t+1}^{w} \right]$$

$$\theta_w \left(\frac{\bar{\pi}_t}{\pi_t^w} \right)^{1-\varepsilon_w} + (1-\theta_w)\,\omega_w \left(\frac{\left(\pi_{t-1}^{w}\right)^{\gamma_w} (\bar{\pi}_t)^{1-\gamma_w}}{\pi_t^w} \right)^{1-\varepsilon_w} + (1-\theta_w)(1-\omega_w) \left(\frac{w_t^*}{w_t} \right)^{1-\varepsilon_w} = 1$$

- Open economy
  - New-Keynesian Phillips curve for imported goods (A.25), (A.26), (A.28)

$$\pi_t^m = \frac{p_t^m}{p_{t-1}^m} \pi_t$$

$$F_{1t}^{m} = \lambda_t M_t \frac{\varepsilon_m}{\varepsilon_m - 1} s_t p_t^{mf} + \beta\theta_m E_t \left[\left(\frac{\bar{\pi}_{t+1}}{\pi_{t+1}^{m}} \right)^{-\varepsilon_m} F_{1t+1}^{m} \right]$$

$$F_{2t}^{m} = \lambda_t M_t + \beta\theta_m E_t \left[\frac{\bar{\pi}_{t+1}}{\pi_{t+1}} \left(\frac{\bar{\pi}_{t+1}}{\pi_{t+1}^{m}} \right)^{-\varepsilon_m} F_{2t+1}^{m} \right]$$

$$\theta_m \left(\frac{\bar{\pi}_t}{\pi_t^m} \right)^{1-\varepsilon_m} + (1-\theta_m)\,\omega_m \left(\frac{\left(\pi_{t-1}^{m}\right)^{\gamma_m} (\bar{\pi}_t)^{1-\gamma_m}}{\pi_t^m} \right)^{1-\varepsilon_m} + (1-\theta_m)(1-\omega_m) \left(\frac{F_{1t}^m}{p_t^m F_{2t}^m} \right)^{1-\varepsilon_m} = 1$$

  - Foreign demand for noncommodity exports (24)

$$X_t^{nc} = \gamma^f \left(\frac{s_t}{p_t^{nc}} \right)^{\phi} Z_t^f$$

  - Interest rate parity (A.31), (35)

$$s_t = E_t \left[\left(s_{t-1} \frac{\pi_t^f}{\pi_t} \right)^{\varkappa} \left(s_{t+1} \frac{r_t^f \left(1 + \kappa_t^f\right)}{R_t} \pi_{t+1} \right)^{1-\varkappa} \right]$$

$$\kappa_t^f = \varsigma \left(\bar{b}^f - b_t^f \right)$$

  - Balance of payments (27)

$$\frac{b_t^f}{r_t^f \left(1 + \kappa_t^f\right)} - b_{t-1}^f \frac{s_t}{s_{t-1}} = \frac{1}{\bar{Y}} \left(p_t^{nc} X_t^{nc} + p_t^{com} X_t^{com} - p_t^m M_t \right)$$

- Monetary policy rule (23)

$$R_t = \rho_r R_{t-1} + (1-\rho_r) \left(\bar{R} + \rho_\pi (\pi_t - \bar{\pi}_t) + \rho_Y \left(\log Y_t - \log \bar{Y}_t \right) \right) + \eta_t^r$$

- Market clearing conditions (32), (33), (34)

$$Z_t = C_t + \iota_i I_t + \iota_x X_t^{nc} + Z_t^{com} + \upsilon_z Z_t$$

$$Y_t = C_t + I_t + X_t^{nc} + X_t^{com} - M_t + \upsilon_y Y_t$$

$$p_t^y Y_t = C_t + p_t^i I_t + p_t^{nc} X_t^{nc} + p_t^{com} X_t^{com} - p_t^m M_t + \upsilon_y p_t^y Y_t$$

$$COM_t = COM_t^d + X_t^{com}$$

- Exogenous processes
  - Processes for shocks

$$\eta_t^r = \varphi_r \eta_{t-1}^r + \xi_t^r$$

$$\log(A_t) = \varphi_a \log(A_{t-1}) + (1-\varphi_a)\log\left(\bar{A}\right) + \xi_t^a$$

$$\log(\eta_t^c) = \varphi_c \log\left(\eta_{t-1}^c\right) + \xi_t^c$$

$$\log\left(Z_t^f\right) = \varphi_{zf}\log\left(Z_{t-1}^f\right) + \left(1-\varphi_{zf}\right)\log\left(\bar{Z}^f\right) + \xi_t^{zf}$$

$$\log\left(p_t^{comf}\right) = \varphi_{comf}\log\left(p_{t-1}^{comf}\right) + \left(1-\varphi_{comf}\right)\log\left(\bar{p}^{comf}\right) + \xi_t^{comf}$$

$$\log\left(r_t^f\right) = \varphi_{rf}\log\left(r_{t-1}^f\right) + \left(1-\varphi_{rf}\right)\log(\bar{r}) + \xi_t^{rf}$$

  - Fixed exogenous prices

$$p_t^i = \iota_i$$

$$p_t^{nc} = \iota_x$$

$$p_t^{mf} = \bar{p}^{mf}$$

  - Targets

$$\bar{\pi}_t = \bar{\pi}$$

$$\log \bar{Y}_t = \varphi_z \log \bar{Y}_{t-1} + (1-\varphi_z)\log\left(\frac{A_t \bar{Y}}{\bar{A}}\right)$$

- Auxiliary expectation terms

$$ex_t^i = E_t\left[\pi_{t+1} p_{t+1}^z \chi_i \left(\frac{I_{t+1}}{I_t} - 1\right)\left(\frac{I_{t+1}}{I_t}\right)^2\right]$$

$$ex_t^{com} = E_t\left[\pi_{t+1} p_{t+1}^{com} \chi_{com} \left(\frac{Z_{t+1}^{com}}{Z_t^{com}} - 1\right)\left(\frac{Z_{t+1}^{com}}{Z_t^{com}}\right)^2\right]$$

D bToTEM parameters

The calibrated values of the parameters for the bToTEM model are summarized in the following two tables.

Parameter	Symbol	Value	Source
Rates			
– real interest rate	$\bar{r}$	1.0076	ToTEM
– discount factor	β	0.9925	ToTEM
– inflation target	$\bar{\pi}$	1.005	ToTEM
– nominal interest rate	$\bar{R}$	1.0126	ToTEM
– ELB on the nominal interest rate	R^{elb}	1.0076	fixed
Output production			
– CES elasticity of substitution	σ	0.5	ToTEM
– CES labor share parameter	δ_l	0.249	calibrated
– CES capital share parameter	δ_k	0.575	calibrated
– CES commodity share parameter	δ_{com}	0.0015	calibrated
– CES import share parameter	δ_m	0.0287	calibrated
– investment adjustment cost	χ_i	20	calibrated
– fixed depreciation rate	d_0	0.0054	ToTEM
– variable depreciation rate	$\bar{d}$	0.0261	ToTEM
– depreciation semielasticity	ρ	4.0931	calibrated
– real investment price	ι_i	1.2698	ToTEM
– real noncommodity export price	ι_x	1.143	ToTEM
– labor productivity	$\bar{A}$	100	normalization
Price setting parameters for consumption			
– probability of indexation	θ	0.75	ToTEM
– RT indexation to past inflation	γ	0.0576	ToTEM
– RT share	ω	0.4819	ToTEM
– elasticity of substitution of consumption goods	ε	11	ToTEM
– Leontieff technology parameter	s_m	0.6	ToTEM
Price setting parameters for imports			
– probability of indexation	θ^m	0.8635	ToTEM
– RT indexation to past inflation	γ^m	0.7358	ToTEM
– RT share	ω^m	0.3	ToTEM
– elasticity of substitution of imports	ε^m	4.4	
Price setting parameters for wages			
– probability of indexation	θ^w	0.5901	ToTEM
– RT indexation to past inflation	γ^w	0.1087	ToTEM
– RT share	ω^w	0.6896	ToTEM
– elasticity of substitution of labor service	ε^w	1.5	ToTEM
Household utility			
– consumption habit	ξ	0.9396	ToTEM
– consumption elasticity of substitution	μ	0.8775	ToTEM
– wage elasticity of labor supply	η	0.0704	ToTEM
Monetary policy			
– interest rate persistence parameter	ρ_r	0.83	ToTEM
– interest rate response to inflation gap	ρ_π	4.12	ToTEM
– interest rate response to output gap	ρ_y	0.4	ToTEM
Other			
– capital premium	κ^k	0.0674	calibrated
– exchange rate persistence parameter	$\varkappa$	0.1585	ToTEM

– foreign commodity price	$\bar{p}^{comf}$	1.6591	ToTEM
– foreign import price	$\bar{p}^{mf}$	1.294	ToTEM
– risk premium response to debt	ς	0.0083	calibrated
– export scale factor	γ^f	18.3113	calibrated
– foreign demand elasticity	ϕ	0.4	calibrated
– elasticity in commodity production	s_z	0.8	calibrated
– land	F	0.1559	calibrated
– share of other components of output	υ_z	0.7651	calibrated
– share of other components of GDP	υ_y	0.311	calibrated
– adjustment cost in commodity production	χ_{com}	16	calibrated
– persistence of potential GDP	φ_z	0.75	calibrated

Table D.1: Calibrated parameters in endogenous model's equations

In Table D.1, we summarize the parameters in the endogenous equations of the model and in Table D.2, we collect the parameters of the exogenous processes for shocks.

Parameter	Symbol	Value	Source
Shock persistence			
– persistence of interest rate shock	φ_r	0.25	ToTEM
– persistence of productivity shock	φ_a	0.9	fixed
– persistence of consumption demand shock	φ_c	0	fixed
– persistence of foreign output shock	φ_{zf}	0.9	fixed
– persistence of foreign commodity price shock	φ_{comf}	0.87	calibrated
– persistence of foreign interest rate shock	φ_{rf}	0.88	calibrated
Shock volatility			
– standard deviation of interest rate shock	σ_r	0.0006	calibrated
– standard deviation of productivity shock	σ_a	0.0067	calibrated
– standard deviation of consumption demand shock	σ_c	0.0001	fixed
– standard deviation of foreign output shock	σ_{zf}	0.0085	calibrated
– standard deviation of foreign commodity price shock	σ_{comf}	0.0796	calibrated
– standard deviation of foreign interest rate shock	σ_{rf}	0.0020	calibrated

Table D.2: Calibrated parameters in exogenous model's equations

E Impulse response functions to foreign shocks

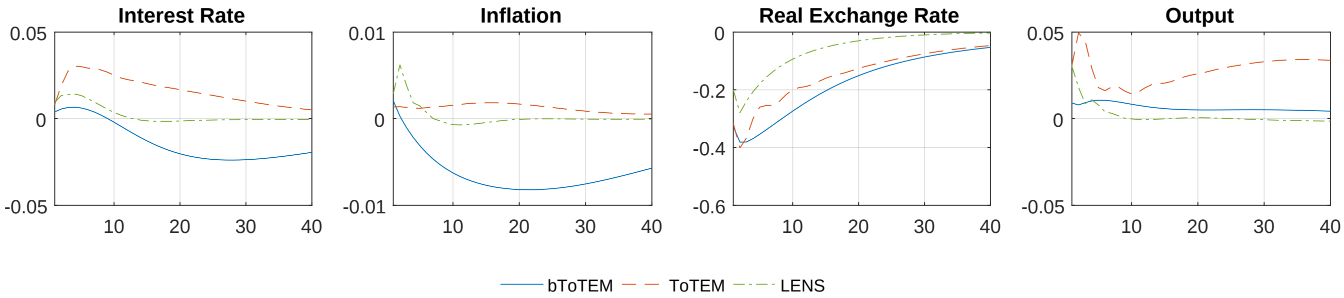

Figure E.1: Impulse response functions: ROW commodity price shock

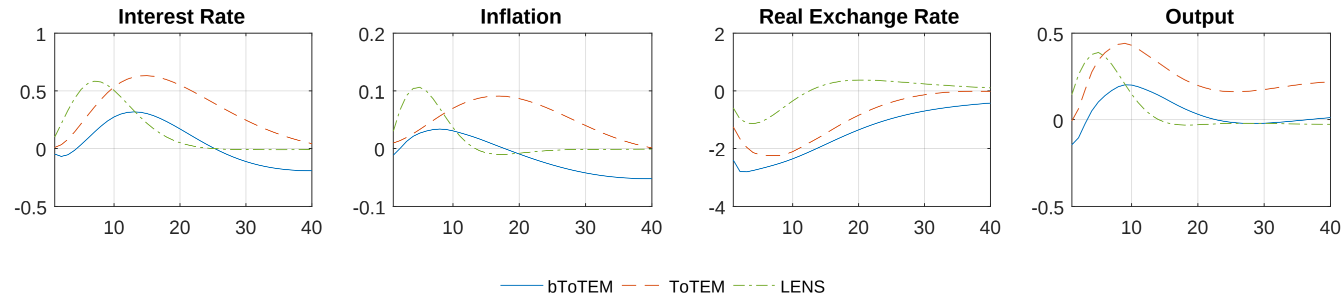

Figure E.2: Impulse response functions: ROW demand shock

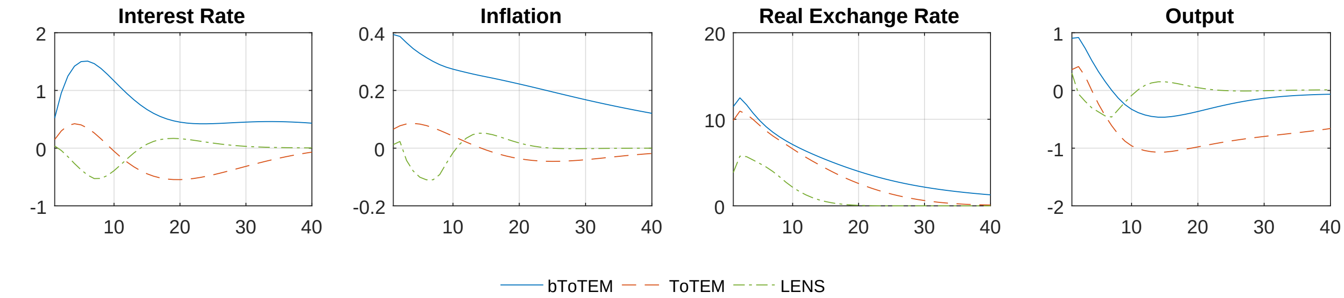

Figure E.3: Impulse response functions: ROW interest rate shock

F Implementation details of the DL solution method

For the purpose of constructing nonlinear global solutions, we split the variables in the bToTEM model into four types:

- exogenous state variables,

$$\mathbf{Z}_t \equiv \left\{A_t, \eta_t^R, \eta_t^c, p_t^{comf}, r_t^f, Z_t^f\right\}, \tag{F.1}$$

- endogenous state variables,

$$\mathbf{S}_t \equiv \left\{C_{t-1}, R_{t-1}, s_{t-1}, \pi_{t-1}, \Delta_{t-1}, w_{t-1}, \pi_{t-1}^w, \Delta_{t-1}^w, p_{t-1}^m, \pi_{t-1}^m, I_{t-1}, Z_{t-1}^{com}, b_{t-1}^f, \bar{Y}_{t-1,} K_{t-1}\right\}, \tag{F.2}$$

- endogenous intertemporal choice variables (these are variables that enter the Euler equation at both t and $t+1$, where a $t+1$ value is a random variable unknown at t),

$$\mathbf{Y}_t \equiv \left\{F_{1t}, F_{2t}, F_{1t}^w, F_{2t}^w, F_{1t}^m, F_{2t}^m, q_t, \lambda_t, s_t, ex_t^i, ex_t^{com}\right\}, \tag{F.3}$$

- and endogenous intratemporal choice variables (these are variables that are determined within the current period t, given the intertemporal choice),

$$\mathbf{X}_t \equiv \left\{ \begin{array}{c} L_t, K_t, I_t, COM_t^d, M_t, u_t, d_t, Z_t^g, Z_t^n, Z_t, C_t, Y_t, \pi_t, rmc_t, \Delta_t, \pi_t^m, \bar{\pi}_t, p_t^m, R_t, p_t^z, w_t, \\ MPK_t, R_t^k, p_t^i, \kappa_t^f, b_t^f, X_t^{nc}, X_t^{com}, COM_t, Z_t^{com}, \pi_t^w, w_t^*, \Delta_t^w, \bar{Y}_t, p_t^{com}, p_t^{nc}, p_t^{mf}, p_t^y \end{array} \right\}. \tag{F.4}$$

Implementation of DL for bToTEM. The DL method is implemented in the context of the bToTEM model as follows:

(Algorithm DL): A global nonlinear DL solution method

Step 0. Initialization

a. Choose simulation length T and fix initial conditions $\mathbf{Z}_0 \equiv \left\{A_0, \eta_0^R, \eta_0^c, p_0^{comf}, r_0^f, Z_0^f\right\}$ and $\mathbf{S}_0$.

b. Draw $\left\{\xi_{t+1}^A, \xi_{t+1}^R, \xi_{t+1}^c, \xi_{t+1}^{comf}, \xi_{t+1}^{rf}, \xi_{t+1}^{Zf}\right\}_{t=0}^{T-1}$ and construct $\mathbf{Z}_t \equiv \left\{A_t, \eta_t^R, \eta_t^c, p_t^{comf}, r_t^f, Z_t^f\right\}_{t=0}^{T}$.

c. Construct perturbation decision function $\widehat{\mathbf{Z}}(\cdot;\mathbf{b}_Z)$, $\widehat{\mathbf{S}}(\cdot;\mathbf{b}_S)$, $\widehat{\mathbf{Y}}(\cdot;\mathbf{b}_Y)$ and $\widehat{\mathbf{X}}(\cdot;\mathbf{b}_X)$, where $\mathbf{b}_Z$, $\mathbf{b}_S$, $\mathbf{b}_Y$ and $\mathbf{b}_X$ are the polynomial coefficients.

d. Use the perturbation solution to produce simulation $\{\mathbf{Y}_t, \mathbf{X}_t, \mathbf{S}_t, \mathbf{Z}_t\}_{t=0}^T$ of $T+1$ observations.

e. Construct a grid for endogenous and exogenous state variables $\{\mathbf{S}_m, \mathbf{Z}_m\}_{m=1,\dots,M}$ by using agglomerative clustering analysis.

f. Choose approximating functions (neural networks) for parameterizing the intertemporal choice: $\mathbf{Y}_t \approx \widehat{\mathbf{Y}}(\cdot;\mathbf{v}_Y)$, where $\mathbf{v}_Y$ is the parameter vector for the global solution method.

g. Use the perturbation solution $\widehat{\mathbf{Y}}(\cdot;\mathbf{b}_Y)$ to construct an initial guess on $\mathbf{v}_Y$.

h. Choose integration nodes, $\left\{\xi_j^A, \xi_j^R, \xi_j^c, \xi_j^{comf}, \xi_j^{rf}, \xi_j^{Zf}\right\}_{j=1,\dots,J}$ and weights, $\{\omega_j\}_{j=1,\dots,J}$.

i. Compute and fix future exogenous states $\mathbf{Z}'_{m,j} \equiv \left\{A_{m,j}, \eta_{m,j}^R, p_{m,j}^c, p_{m,j}^{comf}, r_{m,j}^f, Z_{m,j}^f\right\}_{m=1,\dots,M}$.

Step 1. Updating the intertemporal decision functions

At iteration i, for $m = 1, ..., M$, compute:

a. The intertemporal choice variables $\mathbf{Y}'_m \approx \widehat{\mathbf{Y}}(\mathbf{S}_m, \mathbf{Z}_m; \mathbf{v}_Y)$ (part of this is $\mathbf{S}'_m$).

b. Intratemporal endogenous variables $\mathbf{X}_m$ satisfying the intratemporal choice equations.

c. The intertemporal choice variables in J integration nodes $\mathbf{Y}'_{m,j} \approx \widehat{\mathbf{Y}}\left(\mathbf{S}'_m, \mathbf{Z}'_{m,j}; \mathbf{v}_Y\right)$.

d. Intratemporal endogenous variables $\mathbf{X}_{m,j}$ in J satisfying the intratemporal choice equations.

e. Substitute the results in the intertemporal choice equations and compute $\widehat{\mathbf{Y}}_m$.

f. Find $\mathbf{v}$ that minimizes the distance $\widehat{\mathbf{v}}_Y \equiv \underset{\mathbf{v}}{\operatorname{argmin}} \sum_{m=1}^M \left\|\widehat{\mathbf{Y}}_m - \widehat{\mathbf{Y}}(\mathbf{S}_m, \mathbf{Z}_m; \mathbf{v})\right\|$.

g. Use damping to compute $\mathbf{v}_Y^{(i+1)} = (1-\lambda)\,\mathbf{v}_Y^{(i)} + \lambda \widehat{\mathbf{v}}_Y$, where $\lambda \in (0,1)$ is a damping parameter.

h. Check for convergence and end iteration if $\frac{1}{M} \max \sum_{m=1}^{M} \left|\frac{\mathbf{Y}_m^{(i+1)} - \mathbf{Y}_m^{(i)}}{\mathbf{Y}_m^{(i)}}\right| < \varpi$.

Proceed to the next iteration and iterate on these steps until convergence.

Several comments are in order: To approximate the intertemporal choice functions, we use a three-layer neural network described in the main text. To compute conditional expectations in the intertemporal choice conditions, we use a monomial formula with $2N$ nodes, where $N = 6$ is the number of stochastic shocks; see Judd et al. (2011b) for a description of this formula. In the new Keynesian model studied in Maliar and Maliar (2015), it was possible to derive closed-form expressions for the intratemporal choice, given the intertemporal choice. The bToTEM is more complex and closed-form expressions are infeasible. In this case, we solve for intratemporal choice variables $\mathbf{X}'_m$ using a numerical solver. As for the intratemporal choice variables in the integration nodes $\mathbf{X}'_{m,j}$, we find them either with a numerical solver or by using interpolation of the intratemporal choice decision function $\mathbf{X}'_m$ constructed for the current period using a numerical solver. The damping parameter is set at $\lambda = 0.1$, and the convergence criterion is set at $\varpi = 10^{-7}$.

Our hardware is Intel® Core™ i7-2600 CPU @ 3.400 GHz with RAM 12.0 GB. Our software is written and executed in MATLAB 2016a. We parallelize the computation across four cores. The running time for constructing our global DL solution was about 6 hours; the running time is sensitive to specific choice of the damping parameter λ.

G Accuracy evaluation

	Maximum residual			Average residual		
	Local 1st order	Local 2nd order	Global nn	Local 1st order	Local 2nd order	Global nn
L_t	-2.16	-2.72	-3.60	-2.89	-3.83	-4.65
K_t	-3.60	-4.04	-4.67	-4.34	-5.30	-5.99
I_t	-3.01	-3.38	-3.52	-4.42	-4.50	-4.84
COM_t^d	-2.17	-2.47	-3.68	-2.92	-3.65	-4.43
M_t	-2.16	-2.94	-3.60	-2.89	-4.05	-4.65
u_t	-2.65	-3.20	-3.89	-3.36	-4.43	-5.16
d_t	-2.10	-2.58	-3.37	-2.78	-3.82	-4.64
Z_t^g	-2.29	-3.03	-3.87	-3.05	-4.13	-4.92
Z_t^n	-2.29	-3.03	-3.87	-3.05	-4.13	-4.92
Z_t	-2.29	-3.04	-3.87	-3.06	-4.12	-4.94
C_t	-3.19	-3.11	-4.01	-3.95	-4.23	-5.11
Y_t	-2.58	-3.17	-3.96	-3.24	-3.96	-4.96
π_t	-4.41	-3.84	-4.14	-5.15	-4.92	-4.57
rmc_t	-2.91	-3.15	-4.04	-3.56	-4.30	-5.04
Δ_t	-4.44	-4.86	-5.38	-5.22	-5.32	-6.22
π_t^m	-2.48	-2.70	-3.99	-3.60	-3.76	-5.00
p_t^m	-2.48	-2.73	-4.13	-3.60	-3.78	-4.58
R_t	-3.82	-3.91	-4.27	-4.51	-4.90	-4.74
p_t^z	-2.45	-2.70	-3.56	-3.10	-3.83	-4.57
w_t	-4.13	-4.45	-4.15	-4.81	-5.38	-4.56
p_t^{com}	-2.40	-1.98	-3.05	-3.39	-3.15	-3.97
MPK_t	-2.24	-2.91	-3.34	-3.02	-4.10	-4.53
R_t^k	-2.88	-3.14	-4.27	-4.14	-4.46	-4.74
κ_t^f	-3.57	-2.44	-4.62	-4.65	-3.51	-5.62
b_t^f	-2.02	-2.27	-3.05	-3.05	-3.09	-3.98
X_t^{nc}	-2.80	-2.38	-3.45	-3.79	-3.55	-4.37
X_t^{com}	-1.76	-2.29	-3.18	-2.51	-3.07	-4.41
COM_t	-2.28	-2.54	-3.40	-3.21	-3.28	-4.77
Z_t^{com}	-2.64	-2.37	-3.40	-3.25	-3.43	-4.67
π_t^w	-3.95	-3.96	-4.61	-4.89	-5.03	-5.81
w_t^*	-3.19	-3.10	-3.71	-3.97	-4.19	-4.48
Δ_t^w	-1.44	-2.22	-3.47	-2.52	-3.46	-4.94
F_{1t}	-3.33	-1.71	-2.83	-3.79	-2.92	-3.89
F_{1t}	-3.41	-1.73	-2.91	-3.84	-2.94	-3.74
F_{1t}^w	-1.92	-1.54	-2.37	-2.68	-2.67	-3.60
F_{2t}^w	-3.11	-1.95	-2.95	-4.00	-3.16	-4.15
F_{1t}^m	-2.40	-1.48	-2.63	-2.61	-2.73	-3.87
F_{1t}^m	-2.46	-1.61	-3.05	-2.96	-2.83	-3.91
q_t	-2.47	-2.69	-2.91	-3.89	-4.14	-4.16
λ_t	-2.32	-1.78	-2.72	-3.59	-3.02	-3.87
s_t	-2.40	-1.98	-3.05	-3.39	-3.15	-3.97
Average	-2.75	-2.76	-3.62	-3.58	-3.86	-4.63
Max	-1.44	-1.48	-2.37	-2.51	-2.67	-3.60

Table G.1: Experiment 1. Residuals in the model's equations on the impulse-response path, $log10$ units

We assess the accuracy of solution by constructing unit-free residuals in the model's equations on the simulated paths obtained in our experiments. Our choice of points for accuracy evaluation differs from the two conventional choices in the literature, which are a fixed set of points in a multidimensional hypercube (or hypersphere) and a set of points produced by stochastic simulation; see Kollmann et al. (2011). We choose to focus on the path in the experiments because it is precisely the goal of central bankers to attain a high accuracy of solutions in their policy-relevant experiments (rather than on some hypothetical set of points).

For accuracy evaluation, we use a monomial integration rule with $2N^2+1$ nodes, which is more accurate than monomial rule $2N$ used in the solution procedure, where $N = 6$ is the number of the stochastic shocks; see Judd et al. (2011a) for a detailed description of these integration formulas.

The approximation errors reported in the table are computed over 40 quarters of the first experiment with a negative foreign demand shock. The unit-free residual in each model's equation is expressed in terms of the variable reported in the table: such a residual reflects the difference between the value of that variable produced by the decision function of the corresponding solution method and the value implied by an accurate evaluation of the corresponding model equation, in which case the residuals are loosely interpreted as approximation errors in the corresponding variables.

The resulting unit free residuals in the model's equations are reported in $\log 10$ units. These accuracy units allow for a simple interpretation, namely, "-2" means the size of approximation errors of $10^{-2} = 1$ percent while "-2.5" means approximation errors between 10^{-2} and 10^{-3}, more precisely, we have $10^{-2.5} \approx$ 0.3 percent. The average residuals for the first- and second-degree plain perturbation methods, and the second-degree global method are -3.20, -3.45, and -4.11, respectively, and the maximum residuals are -1.43, -1.44, and -2.09, respectively.