A Tractable Framework for Analyzing a Class of Nonstationary Markov Models*

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Abstract

We consider a class of infinite-horizon dynamic Markov economic models in which the parameters of utility function, production function and transition equations change over time. In such models, the optimal value and decision functions are time-inhomogeneous: they depend not only on state but also on time. We propose a quantitative framework, called *extended function path* (EFP), for calibrating, solving, simulating and estimating such nonstationary Markov models. The EFP framework relies on the turnpike theorem which implies that the finite-horizon solutions asymptotically converge to the infinite-horizon solutions if the time horizon is sufficiently large. The EFP applications include unbalanced stochastic growth models, the entry into and exit from a monetary union, information news, anticipated policy regime switches, deterministic seasonals, among others. Examples of MATLAB code are provided.

JEL classification : C61, C63, C68, E31, E52

Key Words: turnpike theorem; time-inhomogeneous models; nonstationary models; semi-Markov models; unbalanced growth; time varying parameters; trends, anticipated shock; parameter shift; parameter drift; regime switches; stochastic volatility; technological progress; seasonal adjustments; Fair and Taylor method; extended path

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1 Introduction

Dynamic stochastic infinite-horizon models are normally built on the assumption of a stationary (time-homogeneous) environment, namely, it is assumed that the economy's fundamentals such as preferences, technologies and laws of motions for exogenous variables do not change over time. In such models, optimal value and decision functions are also stationary, i.e., they depend only on the economy' state but not on time.

However, actual economies evolve over time, experiencing population growth, technological progress, trends in tastes and habits, policy regime changes, evolution of social and political institutions, etc. Modeling time-dependent features requires the assumption that the parameters of economic models systematically change over time. The resulting models are generally nonstationary (time-inhomogeneous) in the sense that the optimal value and decision functions depend on both state and time. To characterize a solution in such models, we need to construct not just one optimal value and decision functions but an infinitely long sequence (path) of such functions, i.e. a separate set of functions for each period of time.[1] Generally, this is a difficult task!

The literature distinguished a number of special cases in which nonstationary dynamic economic models can be reformulated as stationary ones. Labor augmenting technological progress is a well-known example of a deterministic trend that leads to balanced growth and stationarity in the neoclassical growth model; see King et al. (1988).[2] Time-homogeneous Markov processes are also consistent with stationarity, for example, Markov regime switching models (e.g., Davig and Leeper, 2007, 2009, Farmer et al., 2011 and Foerster et al., 2013) and stochastic volatility models (e.g., Bloom, 2009, Fernández-Villaverde and Rubio-Ramírez, 2010, and Fernández-Villaverde et al. 2016). Finally, anticipated shocks of fixed horizon and periodicity are also consistent with stationarity, including deterministic seasonals (e.g., Barsky and Miron, 1989, Christiano and Todd, 2002, Hansen and Sargent, 1993, 2013) and news shocks (Schmitt-Grohé and Uribe, 2012).

However, many interesting nonstationary models do not admit stationary representations. In particular, deterministic trends typically lead to unbalanced growth, for example, investment-specific technical change (see Krusell et al., 2000); capital-augmenting technological progress (see Acemoglu, 2002, 2003); time trends in the volatility of output and labor-income shares (see Mc Connel and Pérez-Quiros, 2000, and Karabarbounis and Neiman, 2014, respectively), etc. Furthermore, anticipated parameter shifts also lead to time-dependent value and decision functions; for example, anticipated accessions of new members to the European Union (e.g., Garmel et al. 2008), presidential elections with predictable outcomes, credible policy announcements, anticipated legislative changes.

In the paper, we focus on these and other generically nonstationary Markov models.[3] We propose a quantitative framework, called *extended function path* (EFP), which makes it possible to construct a sequence of time-varying decision and value functions for time-inhomogeneous Markov models. The condition that lies in the basis of our construction is the so-called *turn-*

[1] We can also think of these models as ones that contain "time" as an additional state variable.

[2] There are examples of balanced growth models that do not satisfy the restrictions in King et al. (1988) but they are limited; see Maliar and Maliar (2004, 2011), Boppart and Krusell (2016) and Grossman et al. (2017).

[3] A Markov model can be nonstationary (i.e., have no stationary unconditional distribution) even if all the parameters are time-invariant, for example, the unit root and explosive processes. We do not explicitly study these kinds of nonstationarities but focus on time-inhomogeneity of the economic environment.

pike theorem. This condition ensures that a solution to the finite-horizon model provides an arbitrarily close approximation to the infinite-horizon solution if the time horizon is sufficiently large.

The EFP framework has three steps: first, we assume that, in some remote period, the economy becomes stationary and construct the usual stationary Markov solution. Second, given the constructed terminal condition, we solve backward the Bellman or Euler equations to construct a sequence of value and decision functions. Finally, we verify that the turnpike property holds. Although our numerical examples are limited to problems with few state variables, we implement EFP in a way that makes it tractable in large-scale applications. Examples of the MATLAB code are provided.

For a simple optimal growth model, we can characterize the properties of the EFP solution analytically, including its existence, uniqueness and time-inhomogeneous Markov structure. Moreover, we can prove a turnpike theorem that shows uniform convergence of the truncated finite-horizon economy to the corresponding infinite-horizon economy. But for more complex models, analytical characterizations are generally infeasible. In the paper, we advocate a numerical approach to turnpike analysis, namely, we check that during a given number of periods, the constructed finite-horizon numerical approximation is insensitive to the specific terminal condition and terminal date assumed. Such a "numerical" way of verifying the turnpike theorem enlarges greatly a class of tractable nonstationary applications.

We illustrate the EFP methodology in the context of three examples.[4] Our first example is a stylized neoclassical growth model with labor-augmenting technological progress. Such a model can be converted into a stationary one by detrending and solved by any conventional solution method, but EFP makes it possible to solve the model, without detrending. Our second example is an unbalanced growth model with capital-augmenting technological progress which cannot be analyzed by conventional solution methods but which can be easily solved by EFP. Our last example is a version of the new Keynesian model that features the forward guidance puzzle, namely, future events have a nonvanishing impact on today's economy no matter how distant these events are. This example shows the limitations of the EFP analysis: Even though the finite-horizon solution can be constructed, it is not a valid approximation to the infinite-horizon solution if the turnpike theorem does not hold.

The idea of approximating infinite-horizon solutions with finite-horizon solutions is not new to the literature but was introduced and developed in several contexts. First, the turnpike analysis dates back to Dorfman et al. (1958), Brock (1971) and McKenzie (1976); see also Nermuth (1978) for a summary of the earlier literature and for generalizations of Brock's (1971) original results. In particular, there are turnpike theorems for models with time-dependent preferences and technologies; see, e.g. Majumdar and Zilcha (1987), and Mitra and Nyarko (1991). However, the turnpike literature in economics has focused exclusively on the existence results and has never attempted to construct time-dependent solutions in practice.[5] The main novelty of our analysis is that we show how to effectively combine the turnpike analysis with numerical

[4] A working paper version of Maliar et al. (2015) presents a collection of further examples and applications, including growth models with news shocks, regime switches, stochastic volatility, deterministic trend in labor shares and depreciation rates, seasonal fluctuations.

[5] In the optimal control theory, the turnpike analysis was used for numerical analysis of some applications; see Anderson and Kokotovic (1987), Trélat and Zuazua (2015), as well as Zaslavski (2019) for a recent comprehensive reference.

techniques to analyze a challenging class of time-inhomogeneous Markov equilibrium problems that are either not studied in the literature yet or studied under simplifying assumptions.

Furthermore, other solution methods in the literature construct finite-horizon approximations to infinite-horizon problems by (implicitly) relying on the turnpike property, in particular, an extended path (EP) method of Fair and Taylor (1983).[6] The key difference between EP and EFP is that the former constructs a path for variables under one specific realization of shocks (by using certainty equivalence approximation), whereas the latter constructs a path for value or decision functions (by using accurate numerical integration methods). As a result, EFP can accurately solve those models in which the EP's certainty equivalence approach is insufficiently accurate. Furthermore, a simulation of the EFP solutions is cheap unlike the simulation of the EP solutions which requires recomputing the optimal path under each new sequence of shocks.

Finally, there is a literature that studies a transition between two aggregate steady states in heterogeneous-agent economies by constructing a deterministic transition path for aggregate quantities and prices; see, e.g., Conesa and Krueger (1999) and Krueger and Ludwig (2007). The EFP analysis includes but is not equivalent to modeling transition from one steady state to another, in particular, some of our applications do not have steady states (e.g., models with deterministic trends and anticipated shocks do not generally have steady state).

The rest of the paper is as follows: In Section 2, we show analytically the turnpike theorem for a nonstationary growth model. In Section 3, we introduce EFP and show how to verify the turnpike theorem numerically. In Section 4, we assess the performance of EFP in a nonstationary test model with a balanced growth path. In Section 5, we use EFP for analyzing an unbalanced growth model with capital-augmenting technological progress. In Section 6, we discuss the limitations of the EFP framework in the context of the stylized new Keynesian model; finally, in Section 7, we conclude.

2 Verifying the turnpike theorem analytically

We analyze a time-inhomogeneous stochastic growth model in which the parameters can change over time. We show that such a model satisfies the *turnpike theorem*, specifically, the trajectory of the finite-horizon economy converges to that of the infinite-horizon economy as the time horizon increases.

[6] Other related path-solving methods are shooting methods, e.g., Lipton et al. (1980), Atolia and Buffie (2009 a,b), a continuous time analysis of Chen (1999); a parametric path method of Judd (2002); an EP method built on Newton-style solver of Heer and Maußner (2010); a framework for analyzing time-dependent linear rational expectation models of Cagliarini and Kulish (2013); a nonlinear predictive control method for value function iteration of Grüne et al. (2013); refinements of the EP method, e.g., Adjemian and Juillard (2013), Krusell and Smith (2015), and Ajevskis (2017).

2.1 Growth model with time-varying parameters

We consider a stylized stochastic growth model but allow for the case when preferences, technology and laws of motion for exogenous variables change over time,

$$\max_{\{c_t, k_{t+1}\}_{t=0}^T} E_0 \left[\sum_{t=0}^{T} \beta^t u_t (c_t) \right] \tag{1}$$

$$\text{s.t. } c_t + k_{t+1} = (1 - \delta) k_t + f_t (k_t, z_t), \tag{2}$$

$$z_{t+1} = \varphi_t (z_t, \epsilon_{t+1}), \tag{3}$$

where $c_t \geq 0$ and $k_{t+1} \geq 0$ denote consumption and capital, respectively; initial condition (k_0, z_0) is given; $u_t : \mathbb{R}_+ \to \mathbb{R}$, $f_t : \mathbb{R}_+^2 \to \mathbb{R}_+$ and $\varphi_t : \mathbb{R}^2 \to \mathbb{R}$ are time-inhomogeneous utility function, production function and Markov process for exogenous variable z_t, respectively; ϵ_{t+1} is an i.i.d random variable; $\beta \in (0, 1)$ is a discount factor; $\delta \in (0, 1]$ is a depreciation rate; $E_t [\cdot]$ is an operator of expectation, conditional on a t-period information set; and T can be either finite or infinite.

Exogenous variables. In the usual time-homogeneous (stationary) model, the functions $u_t \equiv u$, $f_t \equiv f$ and $\varphi_t \equiv \varphi$ are fixed, time invariant and known to the agent at $t = 0$. For example, if $f (k_t, z_t) = A z_t k_t^{\alpha}$, the agent knows A and α. To construct a time-inhomogeneous model in a parallel manner, we need to fix the sequence of u_t, f_t and φ_t and assume that it is known to the agent at $t = 0$. That is, if $f_t (k_t, z_t) = A_t z_t k_t^{\alpha_t}$, we assume that the agent knows $\{A_t, \alpha_t\}_{t=0}^{\infty}$.

The time-inhomogeneous Markov framework allows us to model a variety of interesting time-dependent scenarios. As an example, consider the technology level A_t. We can assume that A_t can gradually change over time (drifts) or makes sudden jumps (shifts). These changes can be either anticipated or not. In particular, we can have i) technological progress $A_t = A_0 \gamma^t$, where $A_0 > 0$ and γ is the technology growth rate; ii) seasonal fluctuations $A_t = \left\{ \underline{A}, \overline{A}, \underline{A}, \overline{A}, ... \right\}$, where $\underline{A}$, $\overline{A}$ are technology levels in the high and low seasons; iii) news shocks about future levels of A_t; etc.

We can also consider time-dependent scenarios for the parameters of stochastic processes. For example, consider the following process for z_t in (3):

$$\ln z_{t+1} = \rho_t \ln z_t + \sigma_t \epsilon_{t+1}, \tag{4}$$

where $\sigma_t > 0$, $|\rho_t| \leq 1$ and $\epsilon_{t+1} \sim \mathcal{N} (0, 1)$. The process (4) is Markov since the probability distribution $\ln z_{t+1} \sim N (\ln A_t + \rho_t \ln z_t, \sigma_t^2)$ depends only on the current state but not on the history. However, if either the mean $\ln A_t + \rho_t \ln z_t$ or the variance σ_t^2 change over time, then the transition probabilities of $\ln z_{t+1}$ also change over time, i.e., the Markov process is time-inhomogeneous; see Appendix A1 for formal definitions.[7] We can analyze similar time-dependent scenarios for other parameters of the model, including the time-dependent policies.

[7] Mitra and Nyarko (1991) refer to a class of time-inhomogeneous Markov processes as semi-Markov processes because of their similarity to Lévy's (1954) generalization of the Markov renewal process for the case of random arrival times; see Jansen and Manca (2006) for a review of applications of semi-Markov processes in statistics and operation research.

Endogenous variables and the optimal program. A feasible program is a pair of adapted processes $\{c_t, k_{t+1}\}_{t=0}^{T}$ such that, given initial condition (k_0, z_0) and any history $h_T = (\epsilon_0, ..., \epsilon_T)$, reaches a given terminal condition k_{T+1} at T and satisfies $c_t \geq 0$, $k_{t+1} \geq 0$, (2) and (3) for $t = 1, ..., T$. "Adapted" means that the agent does not know future stochastic shocks ϵ's (although she does know the deterministic changes in u_t, f_t and φ_t for all $t \geq 0$).

A feasible program is called *optimal* if it gives higher expected lifetime utility (1) than any other feasible program.

We make standard (strong) assumptions that u_t and f_t are twice continuously differentiable, strictly increasing, strictly quasi-concave and satisfy the Inada conditions for all t. Moreover, we assume that lifetime utility (1) is bounded; see Appendix A2 for a formal description of our assumptions.

The optimal program in the economy (1)–(3) can be characterized by Bellman equations,

$$V_t(k_t, z_t) = \max_{c_t, k_{t+1}} \{u_t(c_t) + \beta E_t [V_{t+1}(k_{t+1}, z_{t+1})]\}, \quad t = 0, 1, ..., T. \tag{5}$$

Also, the interior optimal program satisfies the Euler equations,

$$u_t'(c_t) = \beta E_t \left[u_{t+1}'(c_{t+1})(1 - \delta + f_{t+1}'(k_{t+1}, z_{t+1}))\right], \quad t = 0, 1, ..., T. \tag{6}$$

In our assumptions, we follow Majumdar and Zilcha (1987) and Mitra and Nyarko (1991), except that we assume strict quasi-concavity of the utility and production functions that lead to unique solutions.

2.2 Finite-horizon economy

We first consider a finite-horizon model, $T < \infty$. We know that finite-horizon models are solvable by backward induction from a given terminal condition. To solve such models, we do not need stationarity: the models' parameters (e.g., discount factor, depreciation rate, persistence and volatilities of shocks) can change in every period but backward iteration still works.

Theorem 1 *(Existence and uniqueness of time-inhomogeneous Markov solution). Fix a partial history $h_T = (\epsilon_0, ..., \epsilon_T)$, initial condition (k_0, z_0) and a terminal condition given by a Markov process $K_T(k_T, z_T)$ such that the set of feasible programs is not empty. Then, the optimal program $\{c_t, k_{t+1}\}_{t=0}^{T}$ exists, is unique and is given by a time–inhomogeneous Markov process.*

Proof. The existence of the optimal program $\{c_t, k_{t+1}\}_{t=0}^{T}$ under our assumptions is well known; see, e.g., Theorem 3.1 of Mitra and Nyarko (1991). The uniqueness follows by strict quasi-concavity of the utility and production functions. We are left to check the time-inhomogeneity of the Markov process for the decision functions. We outline the proof by using the Euler equation (6) but a parallel proof can be given via Bellman equation (5); see Majumdar and Zilcha (1987, Theorem 1) for related analysis. Our proof is constructive and follows by backward induction.

Given a T-period (terminal) capital function K_T, we define the capital functions $K_{T-1}, ..., K_0$ in previous periods to satisfy the sequence of the Euler equations. As a first step, we write the Euler equation for period $T - 1$,

$$u_{T-1}'(c_{T-1}) = \beta E_{T-1} \left[u_T'(c_T)(1 - \delta + f_T'(k_T, z_T))\right], \tag{7}$$

where c_{T-1} and c_T are related to k_T and k_{T+1} in periods T and $T-1$ by

$$c_{T-1} = (1-\delta) k_{T-1} + f_{T-1}(k_{T-1}, z_{T-1}) - k_T, \quad (8)$$
$$c_T = (1-\delta) k_T + f_T(k_T, z_T) - k_{T+1}. \quad (9)$$

By our assumptions, $z_T = \varphi_T(z_{T-1}, \epsilon)$ and $k_{T+1} = K_T(k_T, z_T)$ are Markov processes. Combining these assumptions with (7)–(9), we obtain a functional equation that defines k_T for each possible state (k_{T-1}, z_{T-1}), i.e, we obtain an implicitly defined function $k_T = K_{T-1}(k_{T-1}, z_{T-1})$. By proceeding iteratively backward, we construct a sequence of Markov time-dependent capital functions $K_{T-1}(k_{T-1}, z_{T-1}), ..., K_0(k_0, z_0)$ that satisfy (7)–(9) for $t = 0, ..., T-1$ and that matches the terminal function $K_T(k_T, z_T)$. The resulting solution is a time-inhomogeneous Markov process by construction. ■

2.3 Infinite-horizon economy: stationary case

Let us now turn to the infinite-horizon model with $T = \infty$. The literature extensively focuses on the stationary version of (1)–(3) in which preferences, technology and laws of motion for exogenous variables are time homogeneous $u_t = u$, $f_t = f$ and $\varphi_t = \varphi$ for all t. This model has a stationary Markov solution in which value function $V(k_t, z_t)$ and decision functions $k_{t+1} = K(k_t, z_t)$, $c_t = C(k_t, z_t)$ are time invariant and Markov functions that satisfy the stationary versions of the Bellman equation (5) and Euler equation (6), respectively, are

$$V(k_t, z_t) = \max_{c_t, k_{t+1}} \{u(c_t) + \beta E_t[V(k_{t+1}, z_{t+1})]\}, \quad (10)$$

$$u'(c_t) = \beta E_t[u'(c_{t+1})(1 - \delta + f'(k_{t+1}, z_{t+1}))]. \quad (11)$$

The numerical algorithms solve stationary infinite-horizon models by finding fixed point for the value function V and policy function K such that if we substitute them in the right side of the Bellman equation (10) and the Euler equation (11), respectively, we obtain the same functions in the left side of these equations.

However, this solution procedure is not applicable to a time-inhomogeneous version of the model (1)–(3). In such a model, we have different optimal Markov functions V_t, K_t and C_t in each period, and no fixed point exists for such functions or such fixed points are not optimal.

2.4 Infinite-horizon economy: non-stationary case

An alternative we explore in the paper is to approximate an infinite-horizon solution with the corresponding finite-horizon solution. Our analysis is related to the literature on *turnpike theorems*.

2.4.1 Illustration of the turnpike theorem for a model with closed-form solution

Let us first illustrate the turnpike property for a version of the model (1)–(3) that admits a closed-form solution. We specifically assume Cobb-Douglas utility and production functions,

$$u_t(c) = \frac{c^{1-\eta} - 1}{1-\eta}, \quad \text{and} \quad f_t(k, z) = z k^\alpha A_t^{1-\alpha}, \quad (12)$$

where A_t and z_t represent labor-augmenting technological progress and stochastic shock given, respectively, by

$$A_t = A_0 \gamma_A^t \text{ and } \ln z_{t+1} = \rho \ln z_t + \sigma \epsilon_{t+1} \tag{13}$$

where $\gamma_A \geq 1$, $\epsilon_{t+1} \sim \mathcal{N}(0,1)$, $\rho \in (-1,1)$ and $\sigma \in (0,\infty)$.

We consider $\eta = 1$, which leads to a logarithmic utility function, $u_t(c) = \ln c$, and full depreciation of capital, $\delta = 1$. The Bellman equation (5) becomes

$$V_t(k_t) = \max_{k_{t+1}} \left\{ \ln \left(z_t k_t^\alpha A_t^{1-\alpha} - k_{t+1} \right) + \beta E_t \left[V_{t+1}(k_{t+1}) \right] \right\}, \qquad t = 1, ..., T, \tag{14}$$

where we assume $V_{T+1}(k_{T+1}) = 0$ and hence, $k_{T+1} = 0$. It is well known that (14) admits a closed-form solution,

$$k_T = \frac{\alpha\beta}{1+\alpha\beta} z_{T-1} k_{T-1}^\alpha A_{T-1}^{1-\alpha}, \quad k_{T-1} = \frac{\alpha\beta(1+\alpha\beta)}{1+\alpha\beta(1+\alpha\beta)} z_{T-2} k_{T-2}^\alpha A_{T-2}^{1-\alpha}, \quad \text{etc.} \tag{15}$$

In Figure 1, we plot capital trajectories of the economies with finite horizons of $T = 15$ and $T = 25$, as well as of the economy with infinite horizon $T = \infty$. We set the remaining parameters at $\beta = 0.99$, $\alpha = 0.36$, $\rho = 0.95$, $\sigma = 0.01$ and $\gamma_A = 1.01$.

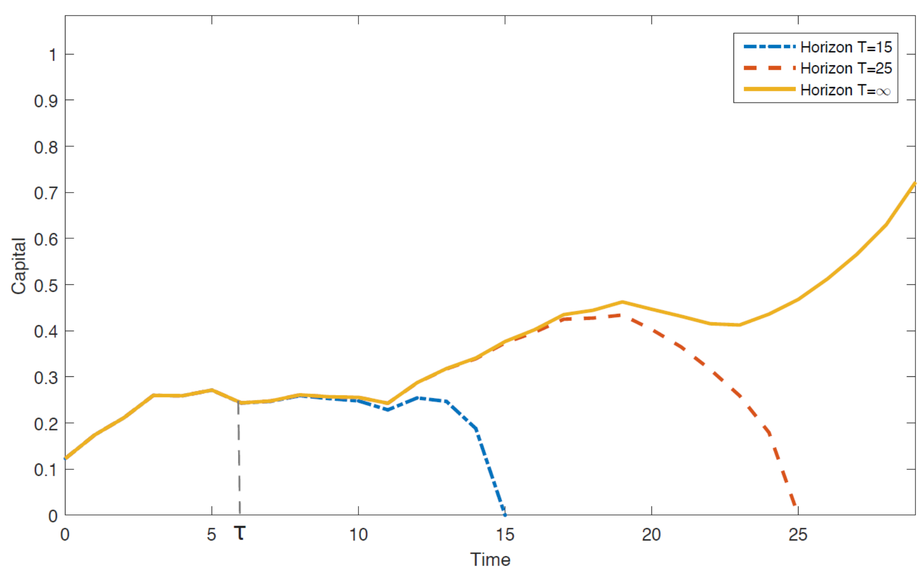

Figure 1. Finite- and infinite-horizon solutions in the growth model.

As we can see, if all three economies start with the same initial capital, they follow a virtually identical path for a long time and diverge only in a close proximity to the terminal date. Therefore, if we are interested in the behavior of infinite-horizon non-stationary economy during some initial number of periods τ, we can accurately approximate the infinite-horizon solution by solving the finite-horizon model. This is precisely what turnpike theorem means.

Figure 1 also helps us understand why this convergence result is called *turnpike theorem*. Turnpike (highway) is often the fastest route between two points even if it is not the shortest one. Specifically, if one drives to some remote destination (e.g., a small town), one typically tries to get on the turnpike as soon as possible, stays on the turnpike for as long as possible and gets off the turnpike as close as possible to the final destination. In the figure, we see the same behavior for the model if we interpret the infinite- and finite-horizon economies as an infinite turnpike and our actual finite destination, respectively.[8]

2.4.2 A formal proof of the turnpike theorem for a general growth model

The turnpike property is not limited to our example with closed-form solution but holds for the growth model (1)–(3) under general utility and production functions. Specifically, we can show that if the time horizon T is sufficiently large, then the finite-horizon solution $\left(k_0^T, ..., k_T^T\right)$ will be within an ε range of the infinite-horizon solution $(k_0^\infty, ..., k_T^\infty)$ during the initial τ periods.

Theorem 2 *(Turnpike theorem). For any real number $\varepsilon > 0$, any natural number τ and any Markov $T-$period terminal condition $K_T(k_T, z_T)$, there exists a threshold terminal date $T^*(\varepsilon, \tau, K_T)$ such that for any $T \geq T^*(\varepsilon, \tau, K_T)$, we have*

$$\left|k_t^\infty - k_t^T\right| < \varepsilon, \quad \text{for all } t \leq \tau, \tag{16}$$

where $k_{t+1}^\infty = K_t^\infty(k_t^\infty, z_t)$ and $k_{t+1}^T = K_t\left(k_t^T, z_t\right)$ are the trajectories in the infinite- and finite-horizon economies, respectively, under given initial condition (k_0, z_0) and partial history $h_T = (\epsilon_0, ..., \epsilon_T)$.

Proof. The proof is shown in Appendix A6, and it relies on three lemmas presented in Appendices A3-A5. First, we construct a limit program of a finite-horizon economy with a terminal condition $k_{T+1} = 0$. Second, we prove the convergence of the optimal program of the T-period stationary economy with an arbitrary terminal capital stock $k_{T+1} = K_T(k_T, z_T)$ to the limiting program of the finite-horizon economy with a zero terminal condition $k_{T+1} = 0$. Finally, we show that the limit program of the finite-horizon economy with zero terminal condition $k_{T+1} = 0$ is also an optimal program for the infinite-horizon nonstationary economy (1)–(3). ■

Remark 1: The above theorem is shown for a fixed history and can be viewed as a sensitivity result. But our analysis can be extended to hold for any history using "almost sure" convergence notion; see, e.g., Majumdar and Zilcha (1987) for such a generalization. The resulting turnpike theorem will state that for all $T \geq T^*(\varepsilon, \tau, K_T)$, the constructed Markov time-inhomogeneous approximation $\left\{k_{t+1}^T\right\}$ is guaranteed to be within a given ε-accuracy range of the true solution $\left\{k_{t+1}^\infty\right\}$ almost surely during the initial τ periods for any history of shocks $h_\infty = (\epsilon_0, \epsilon_1 ...)$.

The first turnpike result dates back to Dorfman et al. (1958) who studied the efficient programs in the von Neumann model of capital accumulation. Their analysis shows that for a long time period and for any initial and terminal conditions, the optimal solution to the model would

[8] We restrict attention to the so-called *early turnpike theorem* which shows that the initial-period decisions functions are insensitive to specific terminal conditions used. There are also medium and late turnpike theorems that focus respectively on the role of the initial and terminal conditions in the properties of the solution; see McKenzie (1976) and Joshi (1997) for a discussion. We do not analyze other turnpike theorems since they are not directly related to the proposed EFP solution framework.

get into the phase of maximal von Neumann growth, i.e. the turnpike. However, the proof of the argument they provided was only valid in a neighborhood of the steady state. The subsequent economic literature provided more general global turnpike theorems; see Brock (1971), McKenzie (1978), Nermuth (1978), Majumdar and Zilch (1987) and Mitra and Nyarko (1991).[9]

However, the turnpike literature in economics has focused exclusively on the existence results and has never attempted to construct time-dependent solutions in practice. The main novelty of our analysis is that we show how to effectively combine the turnpike analysis with numerical techniques to analyze a challenging class of time-inhomogeneous Markov models that are either not studied yet in the literature or studied under simplifying assumptions.

3 Verifying the turnpike theorem numerically: EFP framework

For a simple optimal growth model, it was possible to prove the turnpike theorem analytically. But such analytical proofs are infeasible for more complex and realistic models that are used for applied work. An alternative we offer is to verify the turnpike theorem numerically, namely, we introduce an extended function path (EFP) framework that makes it possible to construct a time-inhomogeneous finite-horizon solutions and to verify that such solutions converge to the infinite-horizon solutions as time horizon increases.

3.1 Markov models with time-varying fundamentals

We analyze two broad classes of nonstationary problems, namely, the optimal control problems and the equilibrium problems. An optimal control time-inhomogeneous Markov problem is characterized by the Bellman equation with a time-dependent value function,

$$V_t\left(x_t, z_t\right) = \max_{x_{t+1}, y_t} \left\{R_t\left(y_t, x_t, z_t\right) + \beta E_t\left[V_{t+1}\left(x_{t+1}, z_{t+1}\right)\right]\right\}, \qquad t = 0, 1, ..., T, \tag{17}$$

where T can be either infinite or finite; $z_t \in H_t^z \subseteq \mathbb{R}^{d_z}$, $x_t \in H_t^x \subseteq \mathbb{R}^{d_x}$ and $y_t \in H_t^y \subseteq \mathbb{R}^{d_y}$ are vectors of exogenous state variables and endogenous state and control variables, respectively; a return function $R_t : H_t^y \times H_t^x \times H_t^z \to \mathbb{R}$ is twice continuously differentiable, strictly increasing, strictly concave, bounded and satisfies the Inada conditions.

An equilibrium time-inhomogeneous Markov problem is characterized by a system of time-dependent Euler and other model's equations,

$$E_t\left[Q_t\left(x_t, z_t, y_t, x_{t+1}, z_{t+1}, y_{t+1}\right)\right] = 0, \qquad t = 0, 1, ..., T, \tag{18}$$

[9] Additionally, the turnpike results are available in the literature that studies optimal control problems. Anderson and Kokotovic (1987) show that a solution to a finite time optimal control problem can be obtained by piecing together solutions to infinite-time problems. TrŽlat and Zuazua (2015) provide general turnpike results that do not rely on any specific assumption on the dynamics of the problem. The recent work of Zaslavski (2019) provides necessary and sufficient conditions for the turnpike property for a broad class of discrete-time optimal control problems for continuous-time infinite-dimensional optimal control problems.

where $Q_t : H_t^x \times H_t^z \times H_t^y \times H_{t+1}^x \times H_{t+1}^z \times H_{t+1}^y \to H_t^Q \subseteq \mathbb{R}^{d_Q}$ is a vector-valued function. In both (17) and (18), a solution satisfies a set of possibly time-dependent constraints:

$$G_t(x_t, z_t, y_t, x_{t+1}) = 0, \qquad t = 0, 1, ..., T, \tag{19}$$

where $G_t : H_t^x \times H_t^z \times H_t^y \times H_{t+1}^x \to \mathbb{R}^G$ is a vector-valued function that is continuously differentiable in x_t, y_t, x_{t+1}; the set $\{(x_t, x_{t+1}) : G_t(x_t, z_t, y_t, x_{t+1}) = 0\}$ is convex and compact. The law of motion for exogenous state variables is given by time-inhomogeneous Markov process $z_{t+1} = Z_t(z_t, \epsilon_{t+1})$, where $Z_t : H_t^z \times H_t^\epsilon \to H_{t+1}^z$ and $\epsilon_t \in H_t^\epsilon \subseteq \mathbb{R}^{d_\epsilon}$ is a vector of independently and identically distributed disturbances. The set of constraints (19) can be generalized to include inequality constraints and the corresponding Karush-Kuhn-Tucker conditions. Some optimal control problems of type (17) can be represented in the form of equilibrium problems of type (18) and vice versa but it is not always the case.

3.2 Turnpike property

We assume that the studied classes of economies satisfy the turnpike property, which we postulate by generalizing the turnpike theorem.

Turnpike property: *For any real number $\varepsilon > 0$, any natural number τ and any Markov T-period terminal condition $X_T(x_T, z_T)$, there exists a threshold terminal date $T^*(\varepsilon, \tau, X_T)$ such that for any $T \geq T^*(\varepsilon, \tau, X_T)$, we have*

$$\left| x_t^\infty - x_t^T \right| < \varepsilon, \quad \text{for all } t \leq \tau, \tag{20}$$

where $x_{t+1}^\infty = X_t^\infty(k_t^\infty, z_t)$ and $x_{t+1}^T = X_t(k_t^T, z_t)$ are time-inhomogeneous Markov trajectories of the infinite- and finite-horizon economies, respectively, under given initial condition (x_0, z_0) and partial history $h_T = (\epsilon_0, ..., \epsilon_T)$.

3.3 EFP framework

EFP aims to accurately approximate a sequence of time-dependent value and decision functions during a given number of periods τ by truncating the infinite-horizon economy. Since EFP in effect solves a finite-horizon model, it makes no difference for the solution procedure whether the parameters are constant or change in every period of time.

Algorithm 1: Extended function path (EFP).
Step 1: Terminal condition. Choose $T \gg \tau$ and construct time-invariant Markov decision functions $X_T(x,z) \equiv X(x,z)$.
Step 2: Backward iteration. Given the terminal condition $X_T \equiv X$, iterate backward on the Bellman or Euler equations to construct a path of time-inhomogeneous Markov decision functions $(X_{T-1},...,X_0)$.
Step 3: Turnpike property. Verify that the initial τ functions $(X_0,...,X_\tau)$ are not sensitive to the choice of time horizon X_T and terminal condition T by analyzing different X_T and T.
Use $(X_0,...,X_\tau)$ as an approximate solution and discard the remaining $T-\tau$ functions $(X_{\tau+1},...,X_T)$.

What determines the accuracy of the EFP approximation? Let us denote the EFP finite-horizon solution by $(\widehat{x}_0,...,\widehat{x}_\tau)$. Then, by a triangle inequality, the supnorm error bound on the EFP approximation is given by:

$$\left|x_t^\infty - \widehat{x}_t^T\right| \leq \left|x_t^\infty - x_t^T\right| + \left|x_t^T - \widehat{x}_t^T\right|, \quad \text{for all } t \leq \tau, \tag{21}$$

where $x_t^\infty = X_t\left(x_t^\infty, z_t\right)$, $x_t^T = X_t\left(x_t^T, z_t\right)$ and $\widehat{x}_t^T = \widehat{X}_t(\widehat{x}_t^T, z_t)$ are the trajectories corresponding to the infinite- and finite-horizon solutions and the EFP approximation, respectively. That is, the EFP approximation error has two components: one is the error $\left|x_t^\infty - x_t^T\right|$ that results from replacing the infinite-horizon problem with the finite-horizon problem, and the other is the error $\left|\widehat{x}_t^T - x_t^T\right|$ that arises because the finite horizon solution itself is approximated numerically. The former error depends on the choice of time-horizon and terminal condition, and it can be made arbitrary small by extending the time horizon T, provided that the turnpike property is satisfied. The latter error depends on the accuracy of numerical techniques used by EFP, such as interpolation, integration, solvers, etc. Since EFP relies on the same techniques as do conventional global solution methods, the standard convergence results apply. For example, Smolyak grids used in our analysis can approximate smooth functions with an arbitrary degree of precision when the approximation level increases; see Judd et al. (2014). Below, we discuss how specific implementation of Steps 1-3 affects the accuracy and computational expense of the EFP method.

Step 1: Terminal condition. In Step 1, we can choose any attainable Markov terminal condition. In particular, we can assume that either the economy converges to a deterministic steady state of some stationary model, or that it reaches a stationary solution with time-invariant Markov value and decision functions, or that it approaches a balanced growth path. To solve stationary or balanced growth models, we can use any conventional projection, perturbation and stochastic simulation methods; see Taylor and Uhlig (1990) and Judd (1998) for reviews of the earlier methods; and see Maliar and Maliar (2014) and Fernández-Villaverde at al. (2016) for reviews of more recent literature. It is even possible to use a trivial (zero) terminal condition by assuming that the life ends at T and by setting all variables to zero, so Step 1 is always implementable. Under the turnpike theorem, a specific terminal condition used has just a negligible effect on the solution in the initial periods provided that the time horizon is large enough. How large the time horizon should be for attaining some given accuracy level does depend on the specific terminal condition used. To increase accuracy and to economize on costs, we must construct a terminal condition that is as close as possible to the infinite-horizon solution at T; in Sections 4 and 5, we explain the construction of the terminal condition in numerical examples.

Step 2: Backward iteration. Conventional projection, perturbation and stochastic simulation methods are designed for solving stationary problems and are not directly suitable for analyzing nonstationary problems. However, the techniques used by these methods can be readily employed as ingredients of EFP. In particular, to approximate value and decision functions, we can use a variety of grids, integration rules, approximation methods, iteration schemes, etc. Furthermore, to solve for a path, we can use any numerical procedure that can solve a system of nonlinear equations, including Newton-style solvers, as well as Gauss-Siedel or Gauss-Jacobi iteration methods. To make EFP tractable in large-scale models, we can use low-cost sparse, simulated, cluster and epsilon-distinguishable-set grids; nonproduct monomial and simulation based integration methods; derivative-free solvers; see Maliar and Maliar (2014) for a survey of these techniques.

Step 2 is equivalent to conventional backward (time) iteration on the Bellman and Euler equations, as is done in case of life-cycle models; see, e.g., Krueger and Kubler (2004) and Hasanhodzic and Kotlikoff (2013); see Ríos-Rull (1999) and Nishiyama and Smetters (2014) for reviews of the literature on life-cycle economies.[10] Time iteration requires the existence of integrals in (17) and (18); see, e.g., Chapter 7 in Stachurski (2009) for a discussion of integrability. Furthermore, optimal problem (17) must have a well defined maximum, while the system in equilibrium problem (18) must be invertible with respect to the next-period state variables x_{t+1}. These are very mild restrictions that are satisfied in virtually any model studied in the related literature.

As a criteria of convergence, we can evaluate the decision functions on low-discrepancy grids or simulated series, as is done for stationary economies; see, e.g., Maliar and Maliar (2014). The difference is that in the stationary case, we must check the convergence of just one time-invariant decision function while in the nonstationary case, we must check the pointwise convergence of the entire path (sequence) of such functions.

Step 3. Turnpike property. If we knew or could somehow guess the exact solution X_T^∞ of the infinite-horizon model at T and use it as a terminal condition, the infinite- and finite-horizon trajectories would coincide up to T. However, the infinite-horizon solution is typically unknown in any period (as this is a solution we are trying to find), so the right terminal condition is also unknown. The turnpike property solves the problem of unknown terminal condition: no

[10] Time iteration is commonly used in the context of dynamic programing methods, as well as some Euler equation methods, e.g., Coleman (1991), Mirman et al. (2008), Malin et al. (2011).

matter what terminal condition X_T we use, the finite-horizon solution $(X_0, ..., X_\tau)$ provides an accurate approximation to the infinite-horizon solution $(X_0^\infty, ..., X_\tau^\infty)$ during the given number of periods τ provided that the time horizon T is sufficiently large. However, this is only true if the turnpike property holds, so Step 3 is a critical ingredient of the EFP analysis.

To test the turnpike property, we construct the EFP solution $(X_0, ..., X_\tau)$ under several different Ts and X_Ts; for each constructed solution $(X_0, ..., X_\tau)$, we simulate the economy's trajectories $(x_0^T, ...x_\tau^T)$ under different initial conditions (x_0, z_0) and different histories h_T; and we evaluate the approximation errors or residuals in the model's equations. Provided that the approximation errors or residuals are small for all Ts and X_Ts, we conclude that the EFP solution converges to a unique limit, i.e., $\lim_{T\to\infty}(X_0^T, ...X_\tau^T) = (X_0^*, ...X_\tau^*)$.

Does our test produce types I and II errors? The former type of errors is not possible: if a model satisfies the turnpike theorem, our test will confirm that the solution is insensitive to a specific terminal condition and time horizon used. The latter type of errors is however possible, i.e, we can erroneously conclude that turnpike theorem holds for some models for which effectively it does not. This is because EFP relies on the assumption that the infinite-horizon solution $(x_0^\infty, ...x_\tau^\infty)$ is equivalent to the limit of the finite-horizon solution $(x_0^*, ...x_\tau^*)$. However, there are models in which this is not the case, in particular, dynamic games. For example, finite-horizon prisoners dilemma has a unique stage equilibrium but the infinite-horizon game has a continuous set of equilibria (folk theorems), and a similar kind of multiplicity is observed for dynamic models with hyperbolic consumers; see Maliar and Maliar (2016) for a discussion. Our test cannot detect this problem: by constructing the EFP solution under different Ts and X_Ts, we only check that the finite-horizon solution has a well-defined unique limit $\lim_{T\to\infty}(x_0^T, ...x_\tau^T) = (x_0^*, ...x_\tau^*)$ but we have no way to check that such a limit is equivalent to infinite-horizon solution $(x_0^\infty, ...x_\tau^\infty)$.

3.4 An example: the EFP analysis of nonstationary growth model

We now use the EFP framework to revisit the optimal growth model studied in Section 2. Below, we elaborate the implementation of Algorithm 1 for that specific model.

Algorithm 1a: Extended function path (EFP) for the growth model.
Step 1: Terminal condition. Choose some $T \gg \tau$ and assume that for $t \geq T$, the economy becomes stationary, i.e., $u_t = u$, $f_t = f$ and $\varphi_t = \varphi$ for all $t \geq T$. Construct a stationary Markov capital function K satisfying: $u'(c) = \beta E\left[u'(c')(1-\delta+f'\left(k', \varphi\left(z, \epsilon'\right)\right))\right]$ $c = (1-\delta)\,k + f\left(k, z\right) - k'$ $c' = (1-\delta)\,k' + f\left(k', \varphi\left(z, \epsilon'\right)\right) - k''$ $k' = K\left(k, z\right)$ and $k'' = K\left(k', \varphi\left(z, \epsilon'\right)\right)$.
Step 2: Backward iteration. Construct a time-inhomogeneous path for capital policy functions $(K_0, ..., K_T)$ that matches the terminal condition $K_T \equiv K$ and that satisfies for $t = 0, ...T-1$: $u'_{t-1}(c_{t-1}) = \beta E_{t-1}\left[u'_t(c_t)(1-\delta+f'_t\left(k_t, \varphi_{t-1}\left(z_{t-1}, \epsilon_t\right)\right))\right]$ $c_{t-1} = (1-\delta)\,k_{t-1} + f_{t-1}\left(k_{t-1}, z_{t-1}\right) - k_t$ $c_t = (1-\delta)\,k_t + f_t\left(k_t, \varphi_{t-1}\left(z_{t-1}, \epsilon_t\right)\right) - k_{t+1}$ $k_t = K_{t-1}\left(k_{t-1}, z_{t-1}\right)$ and $k_{t+1} = K_t\left(k_t, \varphi_{t-1}\left(z_{t-1}, \epsilon_t\right)\right)$.
Step 3: Turnpike property. Draw a set of initial conditions (k_0, z_0) and histories $h_\tau = (\epsilon_0, ..., \epsilon_\tau)$ and use the EFP decision functions $(K_0, ..., K_\tau)$ to simulate the economy's trajectories $(k_0^T, ...k_\tau^T)$. Check that the trajectories converge to a unique limit $\lim_{T\to\infty}(k_0^T, ...k_\tau^T) = (k_0^*, ...k_\tau^*)$ by constructing $(K_0, ..., K_T)$ under different T and K_T.
Use $(K_0, ..., K_\tau)$ as an approximate solution to (17) or (18) and discard the remaining $T-\tau$ functions $(K_{\tau+1}, ..., K_T)$.

Parameterization. We consider the model (1), (2) parameterized by (13) and (12). For all experiments, we fix $\alpha = 0.36$, $\eta = 5$, $\beta = 0.99$, $\delta = 0.025$ and $\rho = 0.95$. The remaining parameters are set in the benchmark case at $\sigma_\epsilon = 0.03$, $\gamma_A = 1.01$ and $T = 200$ but we vary these parameters across experiments.

Step 1: Markov terminal condition. In the studied example, nonstationarity comes from the fact that the economy experiences the economic growth $A_t = A_0\gamma_A^t$. To implement Step 1, we assume that the economic growth stops at T and that the economy becomes stationary, i.e., $A_t = A_T \equiv A$ for all $t \geq T$. To solve the resulting Markov stationary model and construct K_T, we use a conventional projection method based on Smolyak grids as in Judd et al. (2014); this method is tractable in high-dimensional applications.

Step 2: Time-inhomogeneous Markov decision functions. In Step 2, we apply backward iteration: given K_T, we can use the Euler equation to compute K_{T-1} at $T-1$; next, we use K_{T-1} to compute K_{T-2}; we proceed until the entire path $(K_T, ..., K_0)$ is constructed. In Figure 2, we illustrate the resulting sequence of time-inhomogeneous Markov capital decision functions (a function path) produced by Algorithm 1a for the model (1)–(3) with exogenous labor-augmenting technological progress (13).

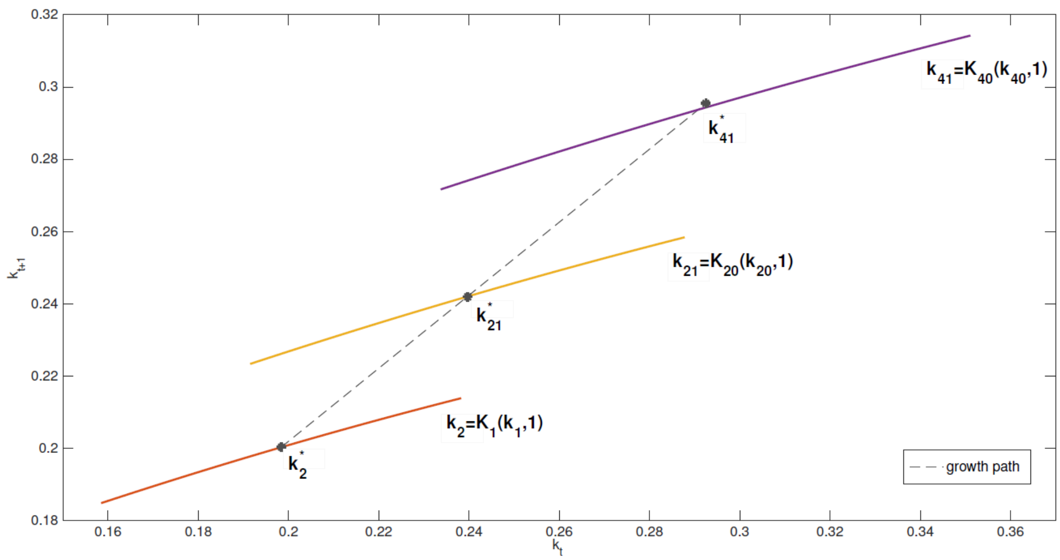

Figure 2. Function path, produced by EFP, for a growth model with technological progress.

As an example, we plot the capital functions for periods 1, 20 and 40 by setting the productivity level equal to one $z = 1$ (i.e., $k_2 = K_1(k_1, 1)$, $k_{21} = K_{20}(k_{20}, 1)$ and $k_{41} = K_{40}(k_{40}, 1)$). In Step 1 of the algorithm, we construct the capital function K_{40} by assuming that the economy becomes stationary in period $T = 40$; in Step 2, we construct a path of the capital functions $(K_0, ...K_{39})$ that matches the corresponding terminal function K_{40}. The domain for capital and the range of the constructed capital function grow at the rate of labor-augmenting technological progress. In Appendix C, we also provide a three-dimensional plot of the capital functions.

Step 3: Numerical verification of the turnpike property. Finally, in Step 3, we verify the turnpike property, i.e., we check that the initial τ decision functions $(K_0, ..., K_\tau)$ are not sensitive to the choice of terminal condition K_T and time horizon T. We implement the test by constructing 100 simulations under random initial conditions (k_0, z_0) and histories $h_T = (\epsilon_0, ..., \epsilon_T)$. We consider two time horizons, $T = 200$ and $T = 400$ and two different terminal conditions, one on the balanced-growth path and the other a solution to the stationary model. The EFP solution proved to be remarkably robust and accurate during the initial periods under all time horizons and terminal conditions considered, in particular, the approximation errors during the first $\tau = 50$ periods do not exceed $10^{-6} = 0.0001\%$. In Section 4, we discuss these accuracy results in details and we compare the EFP solution with those produced by other methods for solving time-inhomogeneous models.

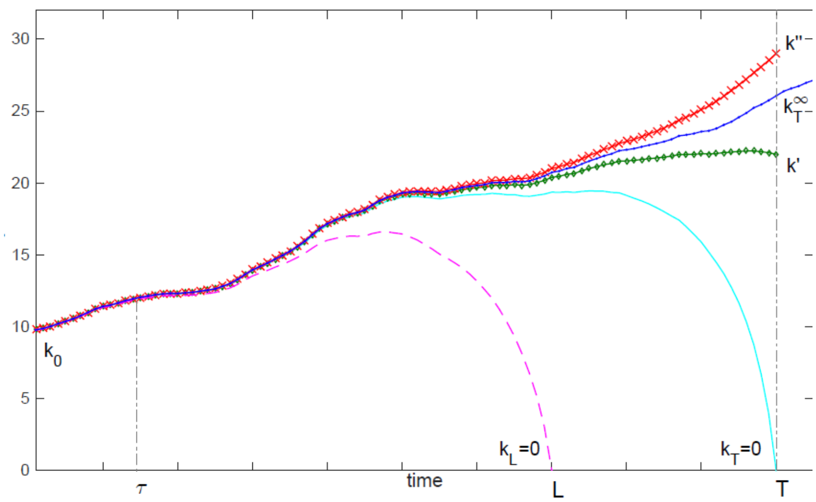

Figure 3. Convergence of the optimal program of T-period stationary economy.

Figure 3 illustrates the turnpike theorem with the graph. For a given initial condition (k_0, z_0) and history of shocks $h_T = (\epsilon_0, ..., \epsilon_T)$, it shows that the initial τ decision functions $(K_0, ..., K_\tau)$ are not sebsitive to various terminal conditions (given by K_L, K_T, k', k'' and K_T^∞). The EFP numerical solution in Figure 3 looks similar to the solution in Figure 1, which we derived analytically. As predicted by the turnpike theorem, the finite-horizon solution converges to the infinite-horizon solution under all terminal conditions considered, however, the convergence is faster under terminal conditions k' and k'', that are located relatively close to the true T-period capital of the nonstationary economy $\{k_T^\infty\}$, than under a zero terminal condition that is located farther away from the true solution. We observe that even though the choice of specific terminal condition plays no role in asymptotic convergence established in the turnpike literature, it can play a critical role in the accuracy and speed of numerical solution methods. To attain the fastest possible convergence, we need to choose the terminal condition K_T^T of the finite-horizon economy to be as close as possible to the T-period capital stock of the infinite-horizon nonstationary economy K_T^∞.

4 EFP versus EP and naive methods

In this section, we assess the quality of the EFP solutions in the optimal growth model (1)–(3) with labor-augmenting technological progress and compare it to solutions produced by other methods. We focus on the growth model that is consistent with balanced growth because in this special case, the nonstationary model can be converted into a stationary model and can be accurately solved by using conventional solution methods; this yields us a high-quality reference solution for comparison.

4.1 Four solution methods

We implement four alternative solution methods which we call *exact*, *EFP*, *Fair and Taylor* and *naive* ones.

i). Exact solution method. We first convert the nonstationarity model into a stationary one using the property of balanced growth; we then accurately solve the stationary model using a Smolyak projection method in line with Krueger and Kubler (2004) and Judd et al. (2014); and we finally recover a solution to the original nonstationary model; see Appendix D for details. The resulting numerical solution is very accurate, namely, the unit-free maximum residuals in the model's equations are of order 10^{-6} on a stochastic simulation of 10,000 observations. We loosely refer to this numerical solution as *exact*, and we use it as a benchmark for comparison.

ii). EFP solution method. EFP constructs a time–inhomogeneous Markov solution to a nonstationary model without converting it into stationary–we follow the steps outlined in Algorithm 1a; see Appendix B for implementation details.

iii). Fair and Taylor (1983) solution method. Fair and Taylor's (1983) method also solves a nonstationary model directly, without converting it into stationary. It constructs a path for the model's variables (not functions!) under one given sequence of shocks by using the certainty equivalence approach for approximating the expectation functions. The implementation of Fair and Taylor's (1983) method is described in Appendix C; for examples of applications of such methods, see, e.g., Chen et al. (2006), Bodenstein et al. (2009), Coibionet al. (2011), Braun and Körber (2011) and Hansen and Imrohoroğlu (2013).

iv). Naive solution method. A naive method replaces a nonstationary model with a sequence of stationary models and solves such models one by one, independently of one another. Similar to EFP, the naive method constructs a path of decision functions for $t = 0, ..., T$ but it differs from EFP in that it neglects the connections between the decision functions in different time periods. A comparison of the EFP and naive solutions allows us to appreciate the importance of anticipatory effects.

The absence of steady state and the deterministic growth path. The studied growing economy has no steady state. However, we can define an analogue of steady state for the growing economy as a solution to an otherwise identical deterministic economy in which the shocks are shut down. We refer to such a solution as growth path, and we denote it by a superscript "$*$". We use the growth path as a sequence of points around which the Smolyak grids are centered. In particular, in Figures 2 and 4, the growth path is shown with a dashed line. In our balanced growth model (12), the growth path can be constructed analytically. Namely, in the detrended economy, the steady state capital is given by $k_0^* \equiv A_0 \left(\frac{\gamma_A^\eta - \beta + \delta\beta}{\alpha\beta} \right)^{1/(\alpha-1)}$, and in the growing economy, it evolves as $k_t^* = k_0^* \gamma_A^t$ for $t = 1, ..., T$; see Appendix D for details. Therefore, we know the exact terminal condition (i.e., the one that coincide with the infinite-horizon solution) for our economy with growth is $k_{T+1}^* = k_0^* \gamma_A^{T+1}$. To assess the role of the terminal condition, we also use another terminal condition that is constructed by assuming that at T, the economy arrives to the steady state with no growth $k_{T+1}^{ss} \equiv A_{T+1}\left(\frac{1-\beta+\delta\beta}{\alpha\beta}\right)^{1/(\alpha-1)}$.

To see how far is T-period steady state terminal condition k_{T+1}^{ss} from the exact growing one k_{T+1}^{*}, we computed the ratio of the two terminal capital stocks,

$$\frac{k_{T+1}^{*}}{k_{T+1}^{ss}} = \left(\frac{\gamma_A^{\eta}-\beta+\delta\beta}{1-\beta+\delta\beta}\right)^{1/(\alpha-1)}.$$

It turned out that this ratio is very different from one under the standard calibration, in particular, with $\eta = 1$, it is 0.67 and with $\eta = 3$, it is 0.38. Thus, by assuming that the economy arrives to the steady state at T, we can overstate the correct terminal capital stock by several times! Using so inaccurate terminal condition requires us to considerably increase the time horizon to make the EFP solutions sufficiently accurate solutions. So, instead of the steady state, we find that it is better to use a terminal condition that leads to a convergence to a nonvanishing growth path – we discuss how to construct such terminal condition in Section 5.

Software and hardware. For all simulations, we use the same initial condition and the same sequence of productivity shocks for all methods considered. Our code is written in MATLAB 2018, and we use a desktop computer with Intel(R) Core(TM) i7-2600 CPU (3.40 GHz) with RAM 12GB.

Comparison results. In the left panel of Figure 4, we plot the growing time-series solutions for the four solution methods, as well as the (steady state) growth path for capital under one specific realization of shocks. In the right panel, we display the time series solutions after detrending the growth path.

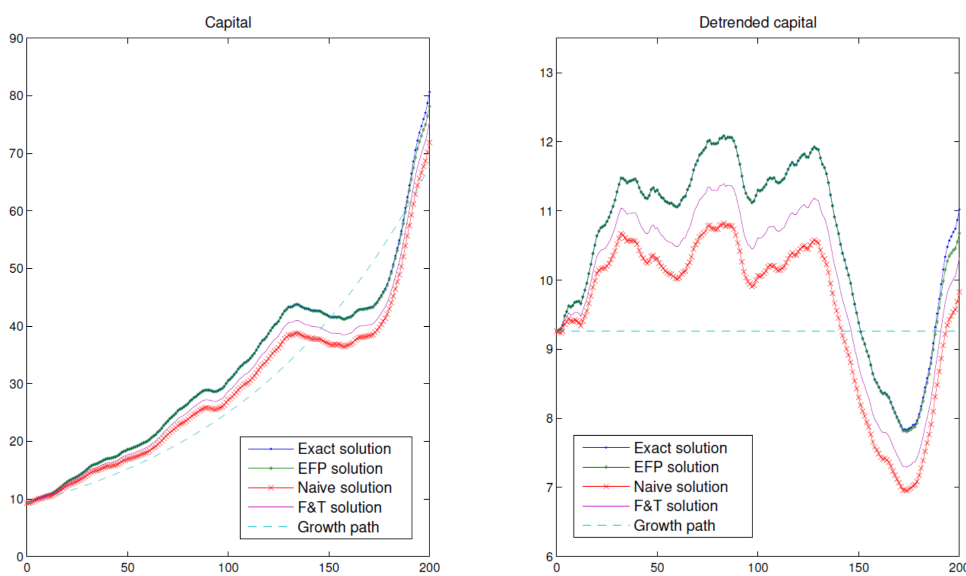

Figure 4. Comparison of the solution methods for the test model with balanced growth.

As is evident from both panels, the EFP and exact solutions are visually indistinguishable except of a small difference at the end of time horizon – the last 10-15 periods. This difference is due to the use of different terminal conditions: in the former case, we assume that the economy becomes stationary (i.e., stops growing) at T, whereas in the latter case, the growth continues forever. If we use the exact solution at T as a terminal condition for the EFP, then the EFP solution would be indistinguishable from the exact solution everywhere in the figure. However, Fair and Taylor's (1983) and naive methods are far less accurate; they produce solutions that are systematically lower than the exact solution everywhere in the figure; and the naive solution is the least accurate of all.

Verifying the turnpike theorem numerically. We next evaluate the accuracy of EFP, Fair and Taylor (1983) and naive solutions by implementing the turnpike test outlined in Steps 3a-3c of Algorithm 1a; see Section 3.4. Specifically, we first simulate each of the four solutions 100 times and we then compute the mean and maximum absolute differences in log 10 units between the exact solution and the remaining three solutions across 100 simulations for the intervals $[0, 50]$, $[0, 100]$, $[0, 150]$, $[0, 175]$, and $[0, 200]$. These statistics show how fast the accuracy of numerical solutions deteriorates, as we approach the terminal period. The accuracy results are reported in Table 1, as well as the time needed for computing and simulating the solution of length T 100 times (in seconds). We observe that in most implementations, the approximation errors of EFP do not exceed $10^{-6} = 0.0001\%$, while the errors produced by Fair and Taylor's (1983) and naive methods can be as large as $10^{-1.13} \approx 7.4\%$ and $10^{-0.89} \approx 12\%$. We discuss these findings below.

4.2 EFP method

In Table 1, we provide the results under three alternative implementations of EFP that illustrate how the properties of the EFP solutions depend on the choices of the terminal condition, K_T, time horizon T and parameter τ.

The role of the terminal condition: better terminal condition gives more accurate solutions. The results in Table 1 show that if we use the balanced-growth terminal condition that is equal to the infinite-horizon solution at T, the EFP approximation is very accurate everywhere independently of τ and T, namely, the difference between the exact and EFP solutions is less than $10^{-6} = 0.0001\%$. In turn, if the terminal condition is given by a solution to a T-period stationary model, the accuracy critically depends on the choice of τ and T, and deteriorates dramatically when the economy approaches the time horizon T, as predicted by the turnpike theorem.

To study how the approximation errors in the tail of the solution depend on the time horizon, terminal condition and model parameters, we also solved the model under $T = \{200, 300, 400, 500\}$ and $\eta = \{1/3, 1, 3\}$; these results are shown in Appendix E. We consider two terminal conditions, one is a T-period stationary economy and the other is a zero-capital assumption. When solving the model for $T = 200$, the maximum errors produced at $\tau = 100$ are about one order of magnitude higher with zero terminal capital than with T-period stationary terminal condition. As we increase T, the errors become smaller independently of the terminal condition. For $T = 300$, the maximum approximation errors vary from $10^{-6} = 0.0001\%$ to

Table 1: Comparison of four solution methods.

	Fair-Taylor (1983) method, $\tau = 1$		Naive method	EFP method $\tau = 1$			EFP method $\tau = 200$		
Terminal condition	Steady state	Steady state	-	Balanced growth	T-period stationary		Balanced growth	T-period stationary	
T	200	400	200	200	200	400	200	200	400
Mean errors across t periods in $\log_{10}$ units									
$t \in [0, 50]$	-1.60	-1.60	-1.36	-7.30	-6.97	-7.15	-7.23	-6.75	-7.01
$t \in [0, 100]$	-1.42	-1.42	-1.19	-7.06	-6.81	-6.98	-7.03	-6.19	-6.81
$t \in [0, 150]$	-1.34	-1.35	-1.11	-6.96	-6.73	-6.91	-6.94	-5.47	-6.73
$t \in [0, 175]$	-1.32	-1.32	-1.09	-6.93	-6.71	-6.89	-6.91	-5.09	-6.70
$t \in [0, 200]$	-1.30	-1.31	-1.07	-6.91	-6.69	-6.87	-6.90	-4.70	-6.68
Maximum errors across t periods in $\log_{10}$ units									
$t \in [0, 50]$	-1.29	-1.29	-1.04	-6.83	-6.63	-6.81	-6.82	-6.01	-6.42
$t \in [0, 100]$	-1.18	-1.18	-0.92	-6.69	-6.42	-6.68	-6.68	-4.39	-5.99
$t \in [0, 150]$	-1.14	-1.14	-0.89	-6.66	-6.39	-6.67	-6.66	-2.89	-5.98
$t \in [0, 175]$	-1.14	-1.13	-0.89	-6.66	-6.40	-6.66	-6.66	-2.10	-5.98
$t \in [0, 200]$	-1.14	-1.13	-0.89	-6.66	-6.37	-6.66	-6.66	-1.45	-5.92
Running time, in seconds									
Solution	1.2(+3)	6.1(+3)	28.9	216.5	8.6(+2)	1.9(+3)	104.9	99.1	225.9
Simulation	-	-	2.6	2.6	2.6	5.8	2.6	2.8	5.7
Total	1.2(+3)	6.1(+3)	31.5	219.2	8.6(+2)	1.9(+3)	107.6	101.9	231.6

Notes: "Mean errors" and "Maximum errors" are, respectively, mean and maximum unit-free absolute difference between the exact solution for capital and the solution delivered by a method in the column. The difference between the solutions is computed across 100 simulations.

$10^{-5} = 0.001\%$. Overall, EFP provides a sufficiently accurate solution for the first 100 periods when we solve the model for $T \geq 250$.

The choice of τ: a trade-off between accuracy and cost. We analyze two different values of τ such as $\tau = 1$ and $\tau = 200$. Under $\tau = 1$, EFP constructs a path of function in the same way as Fair and Taylor's (1983) method constructs a path of variables. First, given K_T, EFP solves for $(K_{T-1}, ..., K_0)$, stores K_0 and discards the rest. Next, given K_{T+1}, EFP solves for $(K_T, ..., K_1)$, stores K_1 and discards the rest. It proceeds for τ steps forward until the path $(K_0, ..., K_\tau)$ is constructed.

As we see from the table, EFP method with $\tau = 1$ is very accurate independently of T and a specific terminal condition used, namely, the EFP and exact solutions again differ by less than $10^{-6} = 0.0001\%$. This result illustrates the implication of the turnpike theorem that the effect of any terminal condition on the very first element of the path $\tau = 1$ is negligible if the time horizon T is sufficiently large.

A shortcoming of the version of EFP with $\tau = 1$ is its high computational expense: the running time under $T = 200$ and $T = 400$ is about 20 and 30 minutes, respectively. The cost is high because we need to recompute entirely a sequence of decision functions each time when we extend the path by one period ahead. Effectively, we recompute the EFP solution T times, and this is what makes it so is costly.

The choice of T: making EFP cheap. Our turnpike theorem suggests a cheaper version of EFP in which we construct a longer path (i.e., we use $\tau > 1$) but we do it just once; the results for this version of the EFP method are provided in the last three columns of Table 1. For $\tau = 200$, the terminal condition plays a critical role in the accuracy of solutions near the tail. Namely, if we use the terminal condition from the T-period stationary economy, and use the time horizon $T = 200$, than the approximation errors near the tail reach $10^{-1.45} \approx 4\%$.

However, the approximation errors are dramatically reduced when the time horizon T increases, as the last column of Table 1 shows. Namely, if we construct a path of length $T = 400$ but use only the first $\tau = 200$ decision functions and discard the remaining path, the solution for the first $\tau = 200$ periods is almost as accurate as that produced under $\tau = 1$. This is true even though the terminal condition from the T-period stationary economy is far away from the exact terminal condition. Importantly, the construction of a longer path is relatively inexpensive: the running time increases from about 2 minutes to 4 minutes when we increase the time horizon from $T = 200$ to $T = 400$, respectively.

Sensitivity analysis. On the basis of the results in Table 1, we advocate a version of EFP that constructs a sufficiently long path $\tau > 1$ by using $T \gg \tau$. We assess the accuracy and cost of this preferred EFP version by using $\tau = 200$ and $T = 400$ under several alternative parameterizations for $\{\eta, \sigma_\epsilon, \gamma_A\}$ such that $\eta \in \{0.1; 1; 5; 10\}$, $\sigma_\epsilon \in \{0.01; 0.03\}$ and $\gamma_A \in \{1; 1.01; 1.05\}$. As a terminal condition, we use decision rules produced by the T-period stationary economy. These sensitivity results are provided in Table 2 of Appendix E.

The accuracy and cost of EFP in these experiments are similar to those reported in Table 1 for the benchmark parameterization. The difference between the exact and EFP solutions varies from $10^{-7} = 0.00001\%$ to $10^{-6} = 0.0001\%$ and the running time varies from 155 to 306 seconds. The exception is the model with a low degree of risk aversion $\eta = 0.1$ for which the running time increases to 842 seconds. (We find that with a low degree of risk aversion, the convergence of EFP is more fragile, so that we had to use a larger degree of damping for iteration, decreasing the speed of convergence).

4.3 Fair and Taylor's (1983) method

As Table 1 shows, EFP dominates the EP method of Fair and Taylor method in both accuracy and cost. Fair and Taylor's (1983) method has relatively low accuracy (namely, its approximation errors are $10^{-1.6} \approx 2.5\%$) because the certainty equivalence approach does not produce sufficiently accurate approximation to conditional expectations under the given parameterization. We find that Fair and Taylor's (1983) method is far more accurate with a smaller variance of shocks and /or smaller degrees of nonlinearities, for example, under $\eta = 1$, $\sigma_\epsilon = 0.01$, $\gamma_A = 1.01$ and $T = 200$, the difference between the exact solution and Fair and Taylor's (1983) solutions is around 0.1% (this experiment is not reported). A comparison of $T = 200$ and $T = 400$ shows that the accuracy of Fair and Taylor's (1983) method cannot be improved by increasing the time horizon.

The high cost of Fair and Taylor's (1983) method is explained by two factors. First, $\tau = 1$ is the only possible choice for Fair and Taylor's (1983) method. To solve for variables of period $t = 0$, this method assumes that productivity shocks are all zeros starting from period $t = 1$, so that the path for $t = 1, ..., T$ has no purpose other than helping to approximate the variables of period $t = 0$. In contrast, EFP can use much longer τs as long as the resulting solution is sufficiently accurate, which reduces the cost.

Second, for Fair and Taylor's (1983) method, the cost of simulating the model is high because the solution and simulation steps are combined together: in order to produce a new simulation, it is necessary to entirely recompute the solution under a different sequence of shocks. In contrast, the simulation cost is low for EFP: after we construct a path of decision functions once, we can use the constructed functions to produce as many simulations as we need under different sequences of shocks. For example, the time that Fair and Taylor's (1983) method needs for computing / simulating 100 solutions is about 30 minutes and 1 hour, respectively (recall that the corresponding times for EFP method are 2 and 4 minutes, respectively).

4.4 Naive method: understanding the importance of anticipatory effects

For the naive method, we report the solution only for $T = 200$ since neither time horizon nor terminal condition are relevant for this method. The performance of the naive method is poor: the difference between the exact and naive solutions can be as large as $10^{-0.89} \approx 12\%$. The naive solution is so inaccurate because the naive method completely neglects anticipatory effects. In each time period t, this method computes a stationary solution by assuming that technology will remain at the levels $A_t = A_0\gamma_A^t$ and $A_{t+1} = A_0\gamma_A^{t+1}$ forever, meanwhile the true nonstationary economy continues to grow. Since the naive agent is "unaware" about the future permanent productivity growth, the expectations of such an agent are systematically more pessimistic than those of the agent who is aware of future productivity growth. It was pointed out by Cooley et al. (1984) that naive-style solution methods are logically inconsistent and contradict to the rational expectation paradigm: agents are unaware about a possibility of parameter changes when they solve their optimization problems, however, they are confronted with parameter changes in simulations. Our analysis suggests that naive solutions are particularly inaccurate in growing economies.

To gain intuition into why the accuracy of the naive method is low and how the expectation about the future can affect today's economy, we perform an additional experiment. We specifically consider a version of the model (1)–(3) with the production function $f_t(k,z) = z_t k_t^{\alpha} A_t$, in which the technology level A_t can take two values, $\underline{A} = 1$ (low) and $\bar{A} = 1.2$ (high). We consider a scenario when the economy starts with $\underline{A}$ at $t = 0$, switches to $\bar{A}$ at $t' = 250$ and then switches back to $\underline{A}$ at $t'' = 550$ (for example, the U.K. joins the EU in 1973 and it exists the EU in 2019). We show the technology profile in the upper panel of Figure 5.

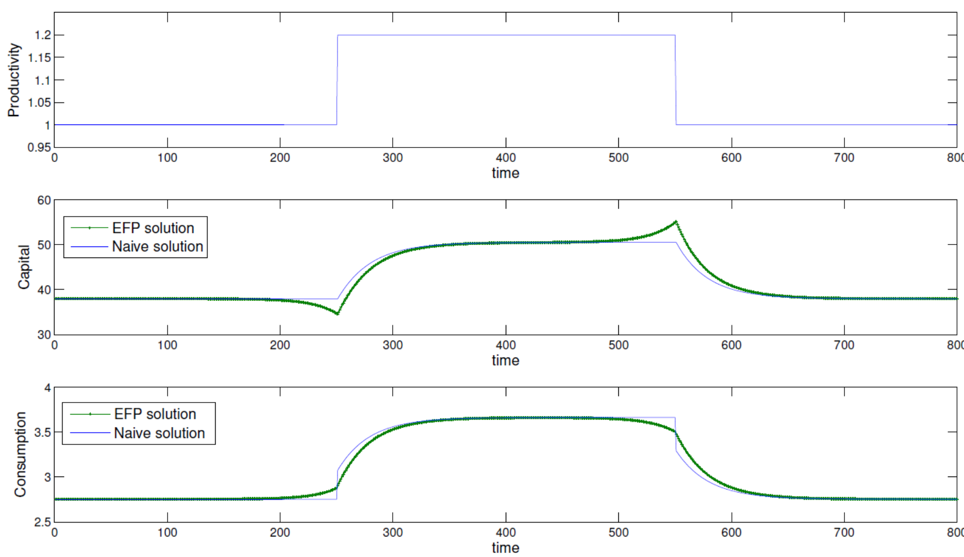

Figure 5. EFP versus naive solutions in the model with parameter shifts.

We parameterize the model by using $T = 900$, $\eta = 1$, $\alpha = 0.36$, $\beta = 0.99$, $\delta = 0.025$, $\rho = 0.95$, $\sigma = 0.01$. To make the anticipatory effects more visible, we shut down the stochastic shocks in simulation by setting $z_t = 1$ for all t.

For a naive agent, regime switches are unexpected. The naive agent believes that the economy will always be in a stationary solution with $\underline{A}$ until the first switch at $t' = 250$, then the agent believes that the economy will always be in a stationary solution with $\bar{A}$ until the second switch at $t'' = 550$ and finally, the agent switches back to the first stationary solution for the rest of the simulation.

In contrast, the EFP method constructs a solution of an informed agent who solves the utility-maximization problem at $t = 0$ knowing the technology profile in Figure 5. Remarkably, under the EFP solution, we observe a strong anticipatory effect: about 50 periods before the switch from $\underline{A}$ and $\bar{A}$ takes place, the agent starts gradually increasing her consumption and decreasing her capital stock in order to bring some part of the benefits from future technological progress to present. When a technology switch actually occurs, it has only a minor effect on consumption. (The tendencies reverse when there is a switch from $\bar{A}$ to $\underline{A}$). Such consumption-smoothing anticipatory effects are entirely absent in the naive solution. Here, unexpected technology shocks lead to large jumps in consumption in the exact moment of technology switches. The difference in the solutions is quantitatively significant under our empirically plausible parameter choice.

Note that anticipated regime changes cannot be effectively approximated by conventional Markov switching models; see, e.g., Sims and Zha (2006), Davig and Leeper (2007, 2009), Farmer et al. (2011), Foerster et al. (2013), etc. In that literature, regime changes come at random and thus, the agents anticipate the possibility of regime change and not the change itself. However, there is a recent literature on Markov chains with time-varying transition probabilities that makes it possible to model the effect of expectation on equilibrium quantities and prices, see, e.g., Bianchi (2019) for a discussion and further references. Also, Schmitt-Grohé and Uribe (2012) propose a perturbation-based approach that deals with anticipated parameter shifts of a fixed time horizons (e.g., shocks that happen each fourth or eight periods) in the context of stationary Markov models. In turn, EFP can handle any combination of unanticipated and anticipated shocks of any periodicity and duration.

5 Using EFP to solve an unbalanced growth model

Real business cycle literature heavily relies on the assumption of labor-augmenting technological progress leading to balanced growth. However, there are empirically relevant models in which growth is unbalanced. For example, Acemoglu (2002) argues that technical change is not always directed to the same fixed factors of production but to those factors of production that give the largest improvement in the efficiency of production.[11] One implication of this argument is that technical change can be directed to either capital or labor depending on the economy's state. Furthermore, Acemoglu (2003) explicitly incorporates capital-augmenting technological progress into a deterministic model of endogenous technical change by allowing for innovations in both capital and labor. Evidence in support of capital-augmenting technical change is provided in, e.g., Klump et al. (2007), and León-Ledesma et al. (2015).[12]

Constant elasticity of substitution production function. In line with this literature, we consider the stochastic growth model (1)–(3) with a constant elasticity of substitution (CES) production function, and we allow for both labor- and capital-augmenting types of technological progress

$$F(k_t,\ell_t) = \left[\alpha(A_{k,t}k_t)^v + (1-\alpha)(A_{\ell,t}\ell_t)^v\right]^{1/v}, \tag{22}$$

where $A_{k,t} = A_{k,0}\gamma^t_{A_k}$; $A_{\ell,t} = A_{\ell,0}\gamma^t_{A_\ell}$; $v \leq 1$; $\alpha \in (0,1)$; and γ_{A_k} and γ_{A_ℓ} are the rates of capital and labour augmenting technological progresses, respectively. We assume that labor is supplied inelastically and normalize it to one, $\ell_t = 1$ for all t, and we denote the corresponding production function by $f(k_t) \equiv F(k_t, 1)$. The Euler equation for the studied model is

$$u'(c_t) = \beta E_t\left[u'(c_{t+1})(1-\delta+\alpha A^v_{k,t+1}(k_{t+1})^{v-1}\left[\alpha(A_{k,t+1}k_{t+1})^v + (1-\alpha)A^v_{\ell,t+1}\right]^{(1-v)/v}\right]. \tag{23}$$

[11] Namely, endogenous technical change is biased toward a relatively more scarce factor when the elasticity of substitution is low (because this factor is relatively more expensive); however, it is biased toward a relatively more abundant factor when the elasticity of substitution is high (because technologies using such a factor have a larger market).

[12] There are other empirically relevant types of technological progress that are inconsistent with balanced growth, for example, investment-specific technological progress considered in Krusell et al. (2000).

The above model is generically nonstationary, specifically, the growth rate of endogenous variables changes over time in an unbalanced manner even if we assume that $A_{k,t}$ and $A_{\ell,t}$ grow at constant growth rates.

A growth path for the economy with unbalanced growth. Our goal is to construct an unbalanced growth path $\{k_t^*\}_{t=0}^{T+1}$ around which the sequence of EFP grids will be centered. We shut down uncertainty by assuming that $z_t = 1$ for all t. First, we construct a terminal condition k_{t+1}^* by assuming that all variables grow at the same rates at T and $T-1$. For this model, it is convenient to target the following two growth rates,

$$\frac{k_{t+2}^*}{k_{t+1}^*} = \frac{k_{t+1}^*}{k_t^*} = \gamma_k \qquad \text{and} \qquad \frac{u'\left(c_{t+1}^*\right)}{u'\left(c_t^*\right)} = \frac{u'\left(c_t^*\right)}{u'\left(c_{t-1}^*\right)} = \gamma_u. \tag{24}$$

With this restriction, the Euler equation (23) written for $T-1$ and T implies

$$1 = \beta\left[\gamma_u(1-\delta+\alpha A_{k,t+1}^v(\gamma_k k_t^*)^{v-1}\left[\alpha(A_{k,t+1}\gamma_k k_t^*)^v + (1-\alpha)A_{\ell,t+1}^v\right]^{(1-v)/v}\right], \tag{25}$$

$$1 = \beta\left[\gamma_u(1-\delta+\alpha A_{k,t}^v(k_t^*)^{v-1}\left[\alpha(A_{k,t}k_t^*)^v + (1-\alpha)A_{\ell,t}^v\right]^{(1-v)/v}\right], \tag{26}$$

where γ_u is determined by the budget constraint (2):

$$\gamma_u = \frac{u'\left[(1-\delta)\gamma_k k_t^* + \left[\alpha(A_{k,t+1}\gamma_k k_t^*)^v + (1-\alpha)A_{\ell,t+1}^v\right]^{1/v} - \gamma_k^2 k_t^*\right]}{u'\left[(1-\delta)k_t^* + \left[\alpha(A_{k,t}k_t^*)^v + (1-\alpha)A_{\ell,t}^v\right]^{1/v} - \gamma_k k_t^*\right]}. \tag{27}$$

Therefore, we obtain a system of three equations (25)-(27) with three unknowns γ_k, γ_u and k_t^*, which we solve numerically. Once the solution is known, we find $k_{t+2}^* = \gamma_k^2 k_t^*$ and $k_{t+1}^* = \gamma_k k_t^*$, calculate c_{t+1}^* from the budget constraint (2) and recover the rest of the growth path $k_{T-1}^*, ..., k_0^*$ by iterating backward on the Euler equation (23).

Results of numerical experiments For numerical experiments, we assume $T = 260$, $\eta = 1$, $\alpha = 0.36$, $\beta = 0.99$, $\delta = 0.025$, $\rho = 0.95$, $\sigma_\epsilon = 0.01$, $v = -0.42$; the last value is taken in line with Antrás (2004) who estimated the elasticity of substitution between capital and labor to be in a range of $[0.641, 0.892]$ that corresponds to $v \in [-0.12, -0.56]$. We solve two models: the model with labor-augmenting progress parameterized by $A_{\ell,0} = 1.1123$, $\gamma_{A_\ell} = 1.0015$ and $A_{k,0} = \gamma_{A_k} = 1$ and the model with capital-augmenting progress parameterized by $A_{k,0} = 1$, $\gamma_{A_k} = 0.9867$ and $A_{\ell,0} = \gamma_{A_\ell} = 1$. The parameters $A_{\ell,0}$, γ_{A_ℓ}, $A_{k,0}$, γ_{A_k} for both models are chosen to approximately match the capital stocks at $t = 0$ and $t = 154$ for the growth paths of capital, so that the cumulative growth is the same for both models over the target period given by $\tau = 154$. To this purpose, we first fix $A_{k,0}$ and γ_{A_k} for the model with capital-augmenting technological change, and we find the values of k_0 and k_{154}. Then, we solve a system of two nonlinear equations (given by a closed-form solution for the model with labor-augmenting technical change) to find the corresponding $A_{\ell,0}$ and γ_{A_ℓ}. The numerical cost of calculating solutions to the model with labor-augmenting and capital-augmenting technical changes are about 1 and 12 minutes, respectively. We implement the turnpike test by verifying that the simulated trajectories are insensitive to the specific time horizon and terminal conditions assumed. Finally, we construct the unit-free residuals in the Euler equation (23), and we find that such residuals do not exceed $10^{-4} = 0.01\%$ across 100 test simulations.

Figure 6 plots the time-series solutions of the models with labour and capital-augmenting technological progresses, as well as the corresponding growth paths.

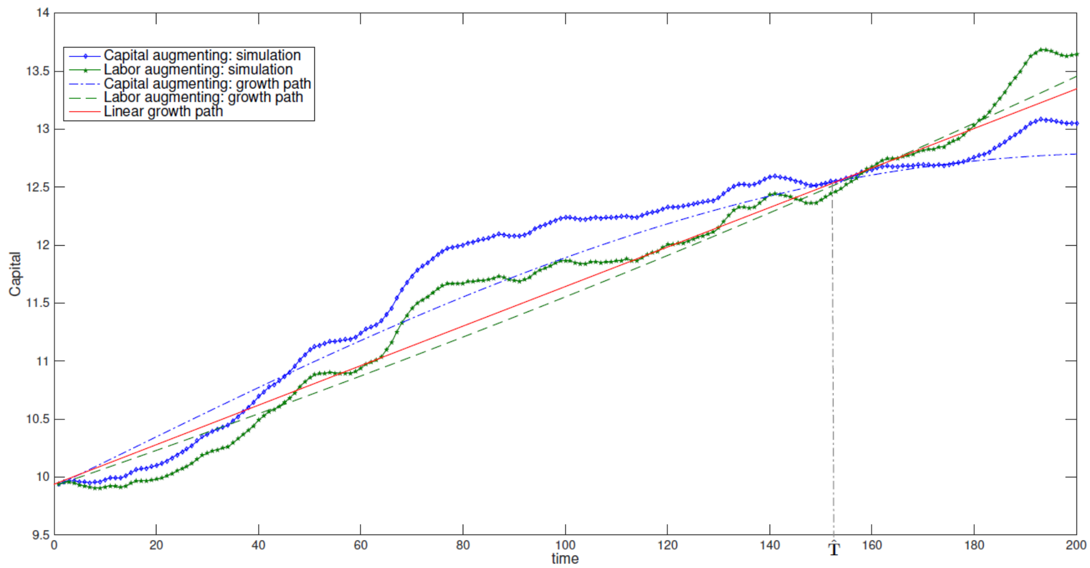

Figure 6. Time-series solution in the model with a CES production function.

The properties of the model with labor-augmenting technological progress are well known. There is an exponential growth path with a constant growth rate and cyclical fluctuations around the growth path. (In the figure, the growth path in the model with labor-augmenting technological progress is situated slightly below the linear growth path shown by a solid line). In contrast, the model with capital-augmenting technological progress is not studied yet in the literature in the presence of uncertainty (to the best of our knowledge). Here, we observe a pronounced concave growth pattern indicating that the rate of return to capital decreases as the economy grows (In the figure, the growth path in the model with capital-augmenting technological progress is situated above the linear growth path shown by a solid line). The cyclical properties of both models look similar (provided that growth is detrended).

6 EFP limitations

We now discuss two limitations of the EFP framework related to the turnpike property and the assumption of Markov structure of the model.

6.1 Turnpike theorem does not always hold

The key assumption behind the EFP analysis is the turnpike property that says that today's decisions are insensitive to events that happen in a distant future. However, not all economic models possess this property. Below, we show a version of the new Keynesian economy in which anticipated future changes in the interest rate have immediate and unrealistically large effects on the current decisions, an implication which is known as a *forward guidance puzzle*; see, e.g., Del Negro et al. (2012), Carlstrom et al. (2015), McKay et. al (2016), Maliar and Taylor (2018).

Consider a stylized new Keynesian framework developed in Woodford (2003) that consists of an IS equation and Phillips curve given by

$$x_t = E_t[x_{t+1}] - \sigma\left(r_t - E_t[\pi_{t+1}] - r_t^n\right), \tag{28}$$
$$\pi_t = \beta E_t[\pi_{t+1}] + \kappa x_t, \tag{29}$$

where x_t, π_t, r_t and r_t^n are the output gap, inflation, nominal interest rate and natural rate of interest, respectively; β and σ are the discount factor and the intertemporal elasticity of substitution, respectively; κ is the slope of the Phillips curve. Suppose that the monetary policy is determined by the following rule

$$r_{t+j} = E_{t+j}[\pi_{t+j+1}] + r_{t+j}^n + \epsilon_{t,t+j}, \tag{30}$$

where $\epsilon_{t,t+j}$ denotes a $t+j$-period shock to the interest rate announced in period t, interpreted as a forward-guidance shock; see Reifschneider and Williams (2000) for a general discussion on monetary policy rules. By applying forward recursion to (28) and by imposing the transversality condition $\lim_{j\to\infty} E_t[x_{t+j}] = 0$, we obtain $x_t = -\sigma\sum_{j=0}^{\infty} E_t\left(r_{t+j} - E_{t+j}[\pi_{t+j+1}] - r_{t+j}^n\right)$ which together with (30) yields

$$x_t = -\sigma\sum_{j=0}^{\infty}\epsilon_{t,t+j}. \tag{31}$$

This result implies that today's shock $\epsilon_{t,t}$ to the interest rate has the same effect as the shock $\epsilon_{t,t+j}$ that happens j years from now. In Figure 7, we show two alternative anticipated future interest rate shocks that happen in period 20 (we assume $\beta = 0.99$, $\kappa = 0.11$ and $\sigma = 1$).

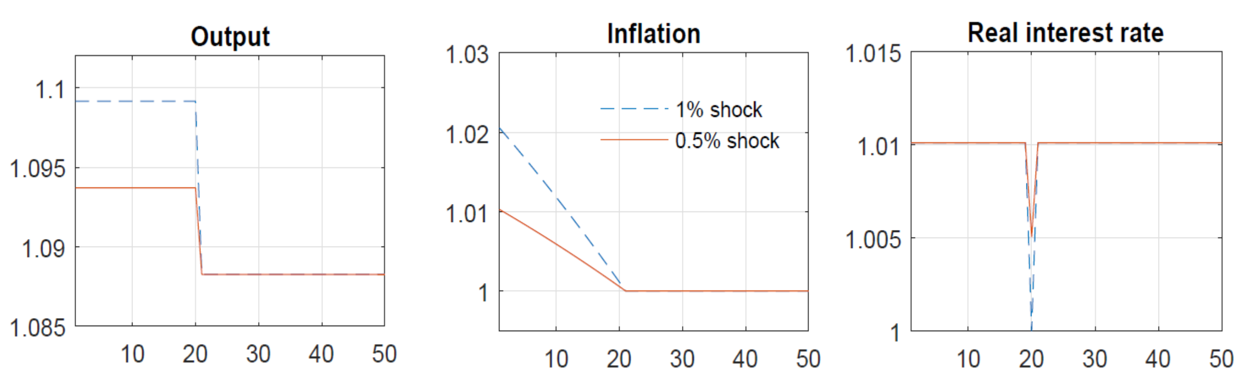

Figure 7. Anticipated future interest-rate shocks in the new Keynesian model.

As we see, decisions in $t = 0, 1, ..., 19$ are dramatically affected by anticipated future interest-rate shocks in period $t = 20$ (such shocks can be viewed as variations in the terminal condition for the interest rate). This example illustrates the failure of numerical verification of the turnpike property in Step 3 of EFP. Specifically, we try out different terminal conditions and different time horizons for the interest rate, and we observe they have a nonvanishing effect on the EFP approximation. We conclude that the turnpike property does not hold and that the EFP methodology is not suitable for analyzing this particular version of the new Keynesian model.[13]

6.2 Not all models are Markov

Another essential assumption of the EFP method is that exogenous variables follow although time-inhomogeneous but still Markov process. If the process for exogenous variables is not Markov, the probability distributions today depend not only on the current state but on the entire historical path of the economy. Hence, the number of states grows exponentially over time and the EFP method becomes intractable. The implications of Markov and history-dependent models may differ dramatically. This fact can be seen by using the example of a new Keynesian model with a zero lower bound on nominal interest rates.

Consider again the IS equation (28) and Phillips curve (29), and assume that the central bank announces that it will peg the nominal interest at zero in some periods j and $j+1$. If the peg is finite and the subsequent terminal condition is consistent with a unique equilibrium, the entire path is uniquely determined; see, e.g., Carlstrom et al. (2015). However, if we construct a Markov solution in which the interest rate is pegged in all states, then equilibrium is indeterminate (this is equivalent to indeterminacy under a perpetual peg; see Galí, 2009).

While the EFP method is not applicable to history-dependent models, there are competing methods that can work with such models. In particular, the EP method of Fair and Taylor (1983) does not rely on the assumption of Markov process and can be used to construct history dependent equilibria, however, its accuracy is limited by the certainty equivalence approach. Adjemian and Juillard (2013) propose a stochastic extended path method that improves on certainty equivalence approach of the baseline Fair and Taylor's (1983) method. They construct and analyze a tree of all possible future paths for exogenous state variables. Although the number of tree branches and paths grows exponentially with the path length, the authors propose a clever way of reducing the cost by restricting attention to paths that have the highest probability of occurrence. However, the implementation of this method is nontrivial, in particular, in models with multiple state variables. Ajevskis (2017) proposes a method in which the certainty equivalent solution a la Fair and Taylor (1983) is improved by incorporating higher order perturbation terms. Furthermore, Krusell and Smith (2015) develop a related numerical method that combines perturbation of distributions and approximate aggregation in line with Krusell and Smith (1998) to solve for a transition path in a multi-region climate change model. Finally, another potentially useful method for analyzing nonstationary applications is a nonlinear particle filter; see, e.g., Fernández-Villaverde et al (2016) for a discussion of this method.

[13]Nonetheless, the version of the model with the forward guidance puzzle is a very special and degenerate case. Maliar and Taylor (2018) show that under empirically relevant parameterizations of the monetary policy rules, new Keynesian models satisfy the turnpike property as well, so that the EFP method can be used.

7 Conclusion

Conventional dynamic programming and Euler-equation methods are designed to solve stationary models by constructing time-invariant rules. In turn, path-solving methods, including Fair and Taylor's (1983) method, can solve nonstationary models by constructing a path for variables. Our analysis combines these two classes of methods by constructing a path for rules. As long as the model satisfies the turnpike property, the path for rules produced by EFP is an accurate approximation of time-varying value and decision functions in the infinite-horizon nonstationary economy. For a simple optimal growth model, the turnpike property can be established analytically but for more complex models, analytical characterizations may be infeasible. As an alternative, we propose to verify the turnpike property numerically by analyzing the sensitivity of the final-horizon EFP solution to terminal conditions and terminal dates. Such "numerical proofs" of turnpike theorems can extend greatly the class of tractable nonstationary models and applications.

Our simple EFP framework has an important value-added in terms of applications that can be analyzed quantitatively. Here are some examples: first, EFP can be applied to solve models with any type of technological progresses (capital, Hicks neutral, investment-specific), as well as any other parameter drifts (e.g., drifts in a depreciation rate, discount factor, utility-function parameters, etc.). Second, EFP can handle any combination of unanticipated and anticipated shocks of any periodicity and duration in a fully nonlinear manner including seasonal adjustments. Third, EFP can be used to analyze models in which volatility has both stochastic and deterministic components. Finally, the EFP framework provides a novel tool for policy analysis: it allows to study time-dependent policies, complementing the mainstream of the literature that focuses on state-dependent policies. In the time-dependent case, a policy maker commits to adopt a new policy on a certain date, independently of the economy's state (e.g., forward guidance about raising the interest rate on a certain future date), whereas in the state-dependent case, a policy maker commits to adopt a new policy when the economy reaches a certain state, independently of the date (e.g., to raise the interest rate when certain economic conditions are met); see Maliar and Taylor (2018) for related forward-guidance policy experiments. Both of these cases are empirically relevant and can be of interest in empirical analysis.

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Online appendices to "A Tractable Framework for Analyzing a Class of Nonstationary Markov Models"

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Appendix A. Turnpike theorem with Markov terminal condition

In this section, we introduce notation, provide several relevant definitions about random processes and elaborate the proof of Theorem 2 (turnpike theorem) formulated in Section 3. The turnpike literature normally assumes a zero terminal capital for the finite-horizon economy, which is a convenient assumption for showing asymptotic convergence results. However, in applications, it is more effective to choose a terminal condition which is as close as possible to the infinite-horizon solution at T. This choice will make the finite-horizon approximation closer to the infinite-horizon solution. (In fact, if we guess the "exact" terminal condition on the infinite-horizon path, then the infinite- and finite-horizon trajectories would coincide). Hence, we show our own version of the turnpike theorem for the model (1)–(3) which holds for an arbitrary Markov terminal condition of the type $k_{T+1} = K_T(k_T, z_T)$, which extends the turnpike literature that focus on a zero terminal condition $k_{T+1} = 0$.

Appendices A1 and A2 contain notations, definitions and preliminaries. The proof of Theorem 2 relies on three lemmas presented in Appendices A3-A5. In Appendix A3, we construct a limit program of a finite-horizon economy with a terminal condition $k_{T+1} = 0$; this construction is standard in the turnpike analysis, see Majumdar and Zilcha (1987), Mitra and Nyarko (1991), Joshi (1997), and it is shown for the sake of completeness. In Appendix A4, we prove a new result about convergence of the optimal program of the T-period stationary economy with an arbitrary terminal capital stock $k_{T+1} = K_T(k_T, z_T)$ to the limiting program of the finite-horizon economy with a zero terminal condition $k_{T+1} = 0$. In Appendix A5, we show that the limit program of the finite-horizon economy with a zero terminal condition $k_{T+1} = 0$ is also an optimal program for the infinite-horizon nonstationary economy; in the proof, we also follow the previous turnpike literature. Finally, in Appendix A6, we combine the results of Appendices A3-A5 to establish the claim of Theorem 2. Thus, our main theoretical contribution is contained in Appendix A4.

Appendix A1. Notation and definitions

Our exposition relies on standard measure theory notation; see, e.g., Stokey and Lucas with Prescott (1989), Santos (1999) and Stachurski (2009). Time is discrete and infinite, $t = 0, 1, ...$. Let $(\Omega, \mathcal{F}, P)$ be a probability space:

a) $\Omega = \Pi_{t=0}^{\infty}\Omega_t$ is a space of sequences $\epsilon \equiv (\epsilon_0, \epsilon_1 ...)$ such that $\epsilon_t \in \Omega_t$ for all t, where Ω_t is a compact metric space endowed with the Borel σ–field $\mathcal{E}_t$. Here, Ω_t is the set of all possible states of the environment at t and $\epsilon_t \in \Omega_t$ is the state of the environment at t.

b) $\mathcal{F}$ is the σ–algebra on Ω generated by cylinder sets of the form $\Pi_{\tau=0}^{\infty} A_\tau$, where $A_\tau \in \mathcal{E}_\tau$ for all τ and $A_\tau = \Omega_\tau$ for all but finitely many τ.

c) P is the probability measure on $(\Omega, \mathcal{F})$.

We denote by $\{\mathcal{F}_t\}$ a filtration on Ω, where $\mathcal{F}_t$ is a sub σ–field of $\mathcal{F}$ induced by a partial history up of environment $h_t = (\epsilon_0, ..., \epsilon_t) \in \Pi_{\tau=0}^{t}\Omega_\tau$ up to period t, i.e., $\mathcal{F}_t$ is generated by cylinder sets of the form $\Pi_{\tau=0}^{t} A_\tau$, where $A_\tau \in \mathcal{E}_\tau$ for all $\tau \leq t$ and $A_\tau = \Omega_\tau$ for $\tau > t$. In particular, we have that $\mathcal{F}_0$ is the course σ–field $\{0, \Omega\}$, and that $\mathcal{F}_\infty = \mathcal{F}$. Furthermore, if Ω consists of either finite or countable states, ϵ is called a *discrete state process* or *chain*; otherwise, it is called a *continuous state process*. Our analysis focuses on continuous state processes, however, can be generalized to chains with minor modifications.

We provide some definitions that will be useful for characterizing random processes; these definitions are standard and closely follow Stokey and Lucas with Prescott (1989, Ch. 8.2).

Definition A1. *(Stochastic process).* A stochastic process on $(\Omega, \mathcal{F}, P)$ is an increasing sequence of σ–algebras $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq ... \subseteq \mathcal{F}$; a measurable space $(Z, \mathcal{Z})$; and a sequence of functions $z_t : \Omega \to Z$ for $t \geq 0$ such that each z_t is $\mathcal{F}_t$ measurable.

Stationarity or time-homogeneity is an assumption that is commonly used in economic literature.

Definition A2. *(Stationary process).* A stochastic process z on $(\Omega, \mathcal{F}, P)$ is called stationary if the unconditional probability measure, given by

$$P_{t+1,...,t+n}(C) = P(\{\epsilon \in \Omega : [z_{t+1}(\epsilon), ..., z_{t+n}(\epsilon)] \in C\}), \tag{32}$$

is independent of t for all $C \in \mathcal{Z}^n$, $t \geq 0$ and $n \geq 1$.

A related notion is stationary (time-homogeneous) transition probabilities. Let us denote by $P_{t+1,...,t+n}(C|z_t = \overline{z}_t, ..., z_0 = \overline{z}_0)$ the probability of the event $\{\epsilon \in \Omega : [z_{t+1}(\epsilon), ..., z_{t+n}(\epsilon)] \in C\}$, given that the event $\{\epsilon \in \Omega : \overline{z}_t = z_t(\epsilon), ..., \overline{z}_0 = z_0(\epsilon)\}$ occurs.

Definition A3. *(Stationary transition probabilities).* A stochastic process z on $(\Omega, \mathcal{F}, P)$ is said to have stationary transition probabilities if the conditional probabilities

$$P_{t+1,...,t+n}(C|z_t = \overline{z}_t, ..., z_0 = \overline{z}_0) \tag{33}$$

are independent of t for all $C \in \mathcal{Z}^n$, $\epsilon \in \Omega$, $t \geq 0$ and $n \geq 1$.

The assumption of stationary transition probabilities (33) implies stationarity (32) if the corresponding unconditional probability measures exist. However, a process can be nonstationary even if transition probabilities are stationary, for example, a unit root process or explosive process is nonstationary; see Stokey and Lucas with Prescott (1989, Ch 8.2) for a related discussion.

In general, $P_{t+1,...,t+n}(C)$ and $P_{t+1,...,t+n}(C|\cdot)$ depend on the entire history of the events up to t (i.e., the stochastic process z_t is measurable with respect to the sub σ–field $\mathcal{F}_t$). However, history-dependent processes are difficult to analyze. The literature distinguishes some special cases in which the dependence on history has relatively simple and tractable form. A well-known case is a class of Markov processes.

Definition A4. *(Time-inhomogeneous Markov process).* A stochastic process z on $(\Omega, \mathcal{F}, P)$ is (first-order) Markov if

$$P_{t+1,...,t+n}(C|z_t = \overline{z}_t, ..., z_0 = \overline{z}_0) = P_{t+1,...,t+n}(C|z_t = \overline{z}_t), \tag{34}$$

for all $C \in \mathcal{Z}^n$, $t \geq 0$ and $n \geq 1$.

The key property of a Markov process is that it is memoryless, namely, all past history $(z_t, ..., z_0)$ is irrelevant for determining the future realizations except of the most recent past z_t. Note that the above definition does not require the Markov process to be time-homogeneous: it allows the functions $P_{t+1,...,t+n}(\cdot)$ to depend on time, as required by our analysis. Finally, if transition probabilities $P_{t+1,...,t+n}(C|z_t = \overline{z}_t)$ are independent of t for any $n \geq 1$, then the Markov process is time-homogeneous. If in addition, there is an unconditional probability measure (32), the resulting Markov process is stationary.

Definition A5. *(Stationary Markov process).* A stochastic process z on $(\Omega, \mathcal{F}, P)$ is called stationary Markov if the unconditional probability measure, given by

$$P_{t+1,...,t+n}(C) = P(\{\epsilon \in \Omega : z_{t+1}(\epsilon) \in C\}), \tag{35}$$

is independent of t for all $C \in \mathcal{Z}^n$, $t \geq 0$ and $n \geq 1$.

Thus, time–homogeneous Markov process is stationary if it has time-homogeneous unconditional probability distribution.

Appendix A2. Infinite-horizon economy

We consider an infinite-horizon nonstationary stochastic growth model in which preferences, technology and laws of motion for exogenous variables change over time. The representative agent solves

$$\max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} E_0 \left[\sum_{t=0}^{\infty} \beta^t u_t(c_t)\right] \tag{36}$$

$$\text{s.t. } c_t + k_{t+1} = (1-\delta) k_t + f_t(k_t, z_t), \tag{37}$$

$$z_{t+1} = \varphi_t(z_t, \epsilon_{t+1}), \tag{38}$$

where $c_t \geq 0$ and $k_t \geq 0$ denote consumption and capital, respectively; initial condition (k_0, z_0) is given; $u_t : \mathbb{R}_+ \rightarrow \mathbb{R}$ and $f_t : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$ and $\varphi_t : \mathbb{R}^2 \rightarrow \mathbb{R}$ are possibly time-dependent utility function, production functions and law of motion for exogenous variable z_t, respectively; the sequence of u_t, f_t and φ_t for $t \geq 0$ is known to the agent in period $t = 0$; ϵ_{t+1} is i.i.d; $\beta \in (0, 1)$ is the discount factor; $\delta \in [0, 1]$ is the depreciation rate; and $E_t[\cdot]$ is an operator of expectation, conditional on a t-period information set.

We make standard assumptions about the utility and production functions that ensure the existence, uniqueness and interiority of a solution. Concerning the utility function u_t, we assume that for each $t \geq 0$:

Assumption 1. *(Utility function). a) u_t is twice continuously differentiable on $\mathbb{R}_+$; b) $u_t' > 0$, i.e., u_t is strictly increasing on $\mathbb{R}_+$, where $u_t' \equiv \frac{\partial u_t}{\partial c}$; c) $u_t'' < 0$, i.e., u_t is strictly concave on $\mathbb{R}_+$, where $u_t'' \equiv \frac{\partial^2 u_t}{\partial c^2}$; and d) u_t satisfies the Inada conditions $\lim_{c \to 0} u_t'(c) = +\infty$ and $\lim_{c \to \infty} u_t'(c) = 0$.*

Concerning the production function f_t, we assume that for each $t \geq 0$:

Assumption 2. *(Production function). a) f_t is twice continuously differentiable on $\mathbb{R}_+^2$, b) $f_t'(k, z) > 0$ for all $k \in \mathbb{R}_+$ and $z \in \mathbb{R}_+$, where $f_t' \equiv \frac{\partial f_t}{\partial k}$, c) $f_t''(k, z) \leq 0$ for all $k \in \mathbb{R}_+$ and $z \in \mathbb{R}_+$, where $f_t'' \equiv \frac{\partial^2 f_t}{\partial k^2}$; and d) f_t satisfies the Inada conditions $\lim_{k \to 0} f_t'(k, z) = +\infty$ and $\lim_{k \to \infty} f_t'(k, z) = 0$ for all $z \in \mathbb{R}_+$.*

We need one more assumption. Let us define a pure capital accumulation process $\left\{k_{t+1}^{\max}\right\}_{t=0}^{\infty}$ by assuming $c_t = 0$ for all t in (37) which for each history $h_t = (z_0, ..., z_t)$, leads to

$$k_{t+1}^{\max} = f_t\left(k_t^{\max}, z_t\right), \tag{39}$$

where $k_0^{\max} \equiv k_0$. We impose an additional joint boundedness restriction on preferences and technology by using the constructed process (39):

Assumption 3. *(Objective function). $E_0\left[\sum_{t=0}^{\infty} \beta^t u_t\left(k_{t+1}^{\max}\right)\right] < \infty$.*

This assumption insures that the objective function (36) is bounded so that its maximum exists. In particular, Assumption 3 holds either (i) when u_t is bounded from above for all t, i.e., $u_t(c) < \infty$ for any $c \geq 0$ or (ii) when f_t is bounded from above for all t, i.e., $f_t(k, z_t) < \infty$ for any $k \geq 0$ and $z_t \in Z_t$. However, it also holds for economies with nonvanishing growth and unbounded utility and production functions as long as $u_t\left(k_{t+1}^{\max}\right)$ does not grow too fast so that the product $\beta^t u_t\left(k_{t+1}^{\max}\right)$ still declines at a sufficiently high rate and the objective function (36) converges to a finite limit.

Definition A6. *(Feasible program).* A feasible program for the economy (36)–(38) is a pair of adapted (t-measurable) processes $\{c_t, k_{t+1}\}_{t=0}^{\infty}$ such that given initial condition k_0, they satisfy $c_t \geq 0$, $k_{t+1} \geq 0$ and (37) for each possible history $h_\infty = (\epsilon_0, \epsilon_1 ...)$.

We denote by $\Im^\infty$ a set of all feasible programs from given initial capital k_0. We next introduce the concept of solution to the model.

Definition A7. *(Optimal program).* A feasible program $\left\{c_t^\infty, k_{t+1}^\infty\right\}_{t=0}^{\infty} \in \Im^\infty$ is called optimal if

$$E_0\left[\sum_{t=0}^{\infty}\beta^t\left\{u_t\left(c_t^{\infty}\right)-u_t\left(c_t\right)\right\}\right]\geq 0 \tag{40}$$

for every feasible process $\{c_t, k_{t+1}\}_{t=0}^{\infty}\in\Im^{\infty}$.

Stochastic models with time-dependent fundamentals are studied in Majumdar and Zilcha (1987), Mitra and Nyarko (1991) and Joshi (1997), among others. The existence results for this class of models have been established in the literature for a general measurable stochastic environment without imposing the restriction of Markov process (38). In particular, this literature shows that, under Assumptions 1-3, there exists an optimal program $\left\{c_t^{\infty}, k_{t+1}^{\infty}\right\}_{t=0}^{\infty}\in\Im^{\infty}$ in the economy (36)–(38), and it is both interior and unique; see Theorem 4.1 in Mitra and Nyarko (1991) and see Theorem 7 in Majumdar and Zilcha (1987). The results of this literature apply to us as well.

Appendix A3. Limit program of finite-horizon economy with a zero terminal capital

In this section, we consider a finite-horizon version of the economy (36)–(38) with a given terminal condition for capital $k_{T+1}=\kappa$. Specifically, we assume that the agent solves

$$\max_{\{c_t,k_{t+1}\}_{t=0}^{T}} E_0\left[\sum_{t=0}^{T}\beta^t u_t\left(c_t\right)\right] \tag{41}$$

$$\text{s.t. (37), (38),} \tag{42}$$

where initial condition (k_0, z_0) and terminal condition $k_{T+1}=\kappa$ are given. We first define feasible programs for the finite-horizon economy.

Definition A8. *(Feasible programs in the finite-horizon economy).* A feasible program in the finite-horizon economy is a pair of adapted (i.e., $\mathcal{F}_t$ measurable for all t) processes $\{c_t, k_{t+1}\}_{t=0}^{T}$ such that given initial condition k_0 and any partial history $h_T=(\epsilon_0,...,\epsilon_T)$, they reach a given terminal condition $k_{T+1}=\kappa$ at T, satisfy $c_t\geq 0$, $k_{t+1}\geq 0$ and (37), (38) for all $t=1,...T$.

In this section, we focus on a finite-horizon economy that reaches a zero terminal condition, $k_{T+1}=0$, at T. We denote by $\Im^{T,0}$ a set of all finite-horizon feasible programs from given initial capital k_0 and any partial history $h_T\equiv(\epsilon_0,...,\epsilon_T)$ that attain given $k_{T+1}=0$ at T. We next introduce the concept of solution for the finite-horizon model.

Definition A9. *(Optimal program in the finite-horizon model).* A feasible finite-horizon program $\left\{c_t^{T,0}, k_{t+1}^{T,0}\right\}_{t=0}^{T}\in\Im^{T,0}$ is called optimal if

$$E_0\left[\sum_{t=0}^{T}\beta^t\left\{u_t(c_t^{T,0})-u_t\left(c_t\right)\right\}\right]\geq 0 \tag{A1}$$

for every feasible process $\{c_t, k_{t+1}\}_{t=0}^{T}\in\Im^{T,0}$.

The existence result for the finite-horizon version of the economy (41), (42) with a zero terminal condition is established in the literature. Namely, under Assumptions A1-A3, there exists an optimal program $\left\{c_t^{T,0}, k_{t+1}^{T,0}\right\}_{t=0}^{T} \in \Im^{T,0}$ and it is both interior and unique. The existence of the optimal program can be shown by using either a Bellman equation approach (see Mitra and Nyarko (1991), Theorem 3.1) or an Euler equation approach (see Majumdar and Zilcha (1987), Theorems 1 and 2).

We next show that under terminal condition $k_{T+1}^{T,0} = k_{T+1} = 0$, the optimal program in the finite-horizon economy (41), (42) has a well-defined limit.

Lemma 1. *A finite-horizon optimal program* $\left\{c_t^{T,0}, k_{t+1}^{T,0}\right\}_{t=0}^{T} \in \Im^{T,0}$ *with a zero terminal condition* $k_{T+1}^{T,0} = 0$ *converges to a limit program* $\left\{c_t^{\lim}, k_{t+1}^{\lim}\right\}_{t=0}^{\infty}$ *when* $T \to \infty$, *i.e.,*

$$k_{t+1}^{\lim} \equiv \lim_{T\to\infty} k_{t+1}^{T,0} \quad \text{and} \quad c_t^{\lim} \equiv \lim_{T\to\infty} c_t^{T,0}, \quad \text{for } t = 0, 1, ... \tag{A2}$$

Proof. The existence of the limit program follows by three arguments (for any history):

i) Extending time horizon from T to $T+1$ increases T-period capital of the finite-horizon optimal program, i.e., $k_{T+1}^{T+1,0} > k_{T+1}^{T,0}$. To see this, note that the model with time horizon T has zero (terminal) capital $k_{T+1}^{T,0} = 0$ at T. When time horizon is extended from T to $T+1$, the model has zero (terminal) capital $k_{T+2}^{T+1,0} = 0$ at $T+1$ but it has strictly positive capital $k_{T+1}^{T+1,0} > 0$ at T; this follows by the Inada conditions–Assumption 1d.

ii) The optimal program for the finite-horizon economy has the following property of monotonicity with respect to the terminal condition: if $\left\{c_t', k_{t+1}'\right\}_{t=0}^{T}$ and $\left\{c_t'', k_{t+1}''\right\}_{t=0}^{T}$ are two optimal programs for the finite-horizon economy with terminal conditions $\kappa' < \kappa''$, then the respective optimal capital choices have the same ranking in each period, i.e., $k_t' \leq k_t''$ for all $t = 1, ...T$. This monotonicity result follows by either Bellman equation programming techniques (see Mitra and Nyarko (1991, Theorem 3.2 and Corollary 3.3)) or Euler equation programming techniques (see Majumdar and Zilcha (1987, Theorem 3)) or lattice programming techniques (see Hopenhayn and Prescott (1992)); see also Joshi (1997, Theorem 1) for generalizations of these results to nonconvex economies. Hence, the stochastic process $\left\{k_{t+1}^{T,0}\right\}_{t=0}^{T}$ shifts up (weakly) in a pointwise manner when T increases to $T+1$, i.e., $k_{t+1}^{T+1,0} \geq k_{t+1}^{T,0}$ for $t \geq 0$.

iii) By construction, the capital program from the optimal program $\left\{c_{t+1}^{T,0}, k_{t+1}^{T,0}\right\}_{t=0}^{T}$ is bounded from above by the capital accumulation process $\left\{0, k_{t+1}^{\max}\right\}_{t=0}^{T}$ defined in (39), i.e., $k_{t+1}^{T,0} \leq k_{t+1}^{\max} < \infty$ for $t \geq 0$. A sequence that is bounded and monotone is known to have a well-defined limit. ■

Appendix A4. Limit program of the T-period stationary economy

We now show that the optimal program of the T-period stationary economy, introduced in Section 4, converges to the same limit program (A2) as the optimal program of the finite-horizon economy (41), (42) with a zero terminal condition. We denote by $\Im^{T,\kappa}$ a set of all feasible finite-horizon programs that attains a terminal condition $\kappa \neq 0$ of the T-period stationary economy. (We assume the same initial capital (k_0, z_0) and the same partial history $h_T \equiv (\epsilon_0, ..., \epsilon_T)$ as those fixed for the finite-horizon economy (41), (42)).

Lemma 2. *The optimal program of the T-period stationary economy $\left\{c_t^{T,\kappa}, k_{t+1}^{T,\kappa}\right\}_{t=0}^{T} \in \Im^{T,\kappa}$ converges to a unique limit program $\{c_t^{\lim}, k_{t+1}^{\lim}\}_{t=0}^{\infty} \in \Im^{\infty}$ defined in (A2) as $T \to \infty$ i.e., for all $t \geq 0$*

$$k_{t+1}^{\lim} \equiv \lim_{T\to\infty} k_{t+1}^{T,\kappa} \quad \text{and} \quad c_t^{\lim} \equiv \lim_{T\to\infty} c_t^{T,\kappa}. \tag{A3}$$

Proof. The proof of the lemma follows by six arguments (for any history).

i). Observe that, by Assumptions 1 and 2, the optimal program of the T-period stationary economy has a positive capital stock $k_{t+1}^{T,\kappa} > 0$ at T (since the terminal capital is generated by the capital decision function of a stationary version of the model), while for the optimal program $\left\{c_t^{T,0}, k_{t+1}^{T,0}\right\}_{t=0}^{T} \in \Im^{T,0}$ of the finite-horizon economy, it is zero by definition, $k_{T+1}^{T,0} = 0$.

ii). The property of monotonicity with respect to terminal condition implies that if $k_{T+1}^{T,\kappa} > k_{T+1}^{T,0}$, then $k_{t+1}^{T,\kappa} \geq k_{t+1}^{T,0}$ for all $t = 1, ..., T$; see our discussion in ii). of the proof to Lemma 1.

iii). Let us fix some $\tau \in \{1, ..., T\}$. We show that up to period τ, the optimal program $\left\{c_t^{T,\kappa}, k_{t+1}^{T,\kappa}\right\}_{t=0}^{\tau}$ does not give higher expected utility than $\left\{c_t^{T,0}, k_{t+1}^{T,0}\right\}_{t=0}^{\tau}$, i.e.,

$$E_0 \left[\sum_{t=0}^{\tau} \beta^t \left\{ u_t\left(c_t^{T,\kappa}\right) - u_t(c_t^{T,0}) \right\} \right] \leq 0. \tag{A4}$$

Toward contradiction, assume that it does, i.e.,

$$E_0 \left[\sum_{t=0}^{\tau} \beta^t \left\{ u_t\left(c_t^{T,\kappa}\right) - u_t(c_t^{T,0}) \right\} \right] > 0. \tag{A5}$$

Then, consider a new process $\{c_t', k_{t+1}'\}_{t=0}^{\tau}$ that follows $\left\{c_t^{T,\kappa}, k_{t+1}^{T,\kappa}\right\}_{t=0}^{T} \in \Im^{T,\kappa}$ up to period $\tau - 1$ and that drops down at τ to match $k_{\tau+1}^{T,0}$ of the finite-horizon program $\left\{c_t^{T,\kappa}, k_{t+1}^{T,\kappa}\right\}_{t=0}^{T} \in \Im^{T,0}$, i.e., $\{c_t', k_{t+1}'\}_{t=0}^{\tau} \equiv \left\{c_t^{T,\kappa}, k_{t+1}^{T,\kappa}\right\}_{t=0}^{\tau-1} \cup \left\{c_\tau^T + k_{\tau+1}^T - k_{\tau+1}^{T,0}, k_{\tau+1}^{T,0}\right\}$. By monotonicity ii). we have $k_{\tau+1}^T - k_{\tau+1}^{T,0} \geq 0$, so that

$$E_0 \left[\sum_{t=0}^{\tau} \beta^t \left\{ u_t\left(c_t'\right) - u_t\left(c_t^{T,\kappa}\right) \right\} \right] =$$

$$= E_0 \left[\beta^\tau \left\{ u_t\left(c_\tau^T + k_{\tau+1}^T - k_{\tau+1}^{T,0}\right) - u_t\left(c_\tau^T\right) \right\} \right] \geq 0, \tag{A6}$$

where the last inequality follows by Assumption 1b of strictly increasing u_t.

iv). By construction $\{c_t', k_{t+1}'\}_{t=0}^{\tau}$ and $\left\{c_t^{T,0}, k_{t+1}^{T,0}\right\}_{t=0}^{\tau}$ reach the same capital $k_{\tau+1}^{T,0}$ at τ. Let us extend the program $\{c_t', k_{t+1}'\}_{t=0}^{\tau}$ to T by assuming that it follows the process $\left\{c_t^{T,0}, k_{t+1}^{T,0}\right\}_{t=0}^{T}$

from the period $\tau+1$ up to T, i.e., $\left\{c'_t, k'_{t+1}\right\}_{t=\tau+1}^{T} \equiv \left\{c_t^{T,0}, k_{t+1}^{T,0}\right\}_{t=\tau+1}^{T}$. Then, we have

$$E_0\left[\sum_{t=0}^{T}\beta^t\left\{u_t\left(c'_t\right)-u_t(c_t^{T,0})\right\}\right] = E_0\left[\sum_{t=0}^{\tau}\beta^t\left\{u_t\left(c'_t\right)-u_t(c_t^{T,0})\right\}\right]$$
$$\geq E_0\left[\sum_{t=0}^{\tau}\beta^t\left\{u_t\left(c_t^{T,\kappa}\right)-u_t\left(c_t^{T,0}\right)\right\}\right] > 0, \quad \text{(A7)}$$

where the last two inequalities follow by result (A6) and assumption (A5), respectively. Thus, we obtain a contradiction: The constructed program $\left\{c'_t, k'_{t+1}\right\}_{t=0}^{T} \in \Im^{T,0}$ is feasible in the finite-horizon economy with a zero terminal condition, $k'_{T+1}=0$, and it gives strictly higher expected utility than the optimal program $\left\{c_t^{T,0}, k_{t+1}^{T,0}\right\}_{t=0}^{T} \in \Im^{T,0}$ in that economy.

v). Holding τ fixed, we compute the limit of (A4) by letting T go to infinity:

$$\lim_{T\to\infty} E_0\left[\sum_{t=0}^{\tau}\beta^t\left\{u_t\left(c_t^{T,\kappa}\right)-u_t(c_t^{T,0})\right\}\right] =$$
$$\lim_{T\to\infty} E_0\left[\sum_{t=0}^{\tau}\beta^t u_t\left(c_t^{T,\kappa}\right)\right] - E_0\left[\sum_{t=0}^{\tau}\beta^t u_t\left(c_t^{\lim}\right)\right] \leq 0. \quad \text{(A8)}$$

vi). The last inequality implies that for any $\tau \geq 1$, the limit program $\left\{c_t^{\lim}, k_{t+1}^{\lim}\right\}_{t=0}^{\infty} \in \Im^{\infty}$ of the finite-horizon economy $\left\{c_t^{T,0}, k_{t+1}^{T,0}\right\}_{t=0}^{T} \in \Im^{T,0}$ with a zero terminal condition $k_T^{T,0}=0$ gives at least as high expected utility as the optimal limit program $\left\{c_t^{T,\kappa}, k_{t+1}^{T,\kappa}\right\}_{t=0}^{T} \in \Im^{T,\kappa}$ of the T-period stationary economy. Since Assumptions 1 and 2 imply that the optimal program is unique, we conclude that $\left\{c_t^{\lim}, k_{t+1}^{\lim}\right\}_{t=0}^{\infty} \in \Im^{\infty}$ defined in (A2) is a unique limit of the optimal program $\left\{c_t^{T,\kappa}, k_{t+1}^{T,\kappa}\right\}_{t=0}^{T} \in \Im^{T,\kappa}$ of the T-period stationary economy. ■

Appendix A5. Convergence of the finite-horizon economy to the infinite-horizon economy

We now show a connection between the optimal programs of the finite-horizon and infinite-horizon economies. Namely, we show that the finite-horizon economy (41), (42) with a zero terminal condition $k_{T+1}^{T,0}=0$ converges to the nonstationary infinite-horizon economy (36)–(38) as $T\to\infty$ provided that we fix the same initial condition k_0 and partial history $h_T=(\epsilon_0,...,\epsilon_T)$ for both economies.

Lemma 3. *The limit program* $\left\{c_t^{\lim}, k_{t+1}^{\lim}\right\}_{t=0}^{\infty}$ *is a unique optimal program* $\left\{c_t^{\infty}, k_{t+1}^{\infty}\right\}_{t=0}^{\infty} \in \Im^{\infty}$ *in the infinite-horizon nonstationary economy (36)–(38).*

Proof. We prove this lemma by contradiction. We use the arguments that are similar to those used in the proof of Lemma 2.

i). Toward contradiction, assume that $\left\{c_t^{\lim}, k_{t+1}^{\lim}\right\}_{t=0}^{\infty}$ is not an optimal program of the infinite-horizon economy $\left\{c_t^{\infty}, k_{t+1}^{\infty}\right\}_{t=0}^{\infty} \in \Im^{\infty}$. By definition of limit, there exists a real number

$\varepsilon > 0$ and a subsequence of natural numbers $\{T_1, T_2, ...\} \subseteq \{0, 1, ...\}$ such that $\left\{c_t^{\infty}, k_{t+1}^{\infty}\right\}_{t=0}^{\infty} \in \Im^{\infty}$ gives strictly higher expected utility than the limit program of the finite-horizon economy $\left\{c_t^{\lim}, k_{t+1}^{\lim}\right\}_{t=0}^{\infty}$, i.e.,

$$E_0\left[\sum_{t=0}^{T_n} \beta^t \left\{u_t\left(c_t^{\infty}\right) - u_t(c_t^{\lim})\right\}\right] > \varepsilon \text{ for all } T_n \in \{T_1, T_2, ...\}. \tag{A9}$$

ii). Let us fix some $T_n \in \{T_1, T_2, ...\}$ and consider any finite $T \geq T_n$. Assumptions 1 and 2 imply that $k_{T+1}^{\infty} > 0$ while $k_{T+1}^{T,0} = 0$ by definition of the finite-horizon economy with a zero terminal condition. The monotonicity of the optimal program with respect to a terminal condition implies that if $k_{T+1}^{\infty} > k_{T+1}^{T,0}$, then $k_{t+1}^{\infty} \geq k_{t+1}^{T,0}$ for all $t = 1, ..., T$; see our discussion in ii). of the proof of Lemma 1.

iii). Following the arguments in iii). of the proof of Lemma 2, we can show that up to period T_n, the optimal program $\left\{c_t^{\infty}, k_{t+1}^{\infty}\right\}_{t=0}^{T_n}$ does not give higher expected utility than $\left\{c_t^{T,0}, k_{t+1}^{T,0}\right\}_{t=0}^{T_n}$, i.e.,

$$E_0\left[\sum_{t=0}^{T_n} \beta^t \left\{u_t\left(c_t^{\infty}\right) - u_t(c_t^{T,0})\right\}\right] \leq 0 \text{ for all } T_n. \tag{A10}$$

iv). Holding T_n fixed, we compute the limit of (A10) by letting T go to infinity:

$$\lim_{T\to\infty} E_0\left[\sum_{t=0}^{T_n} \beta^t \left\{u_t\left(c_t^{\infty}\right) - u_t(c_t^{T,0})\right\}\right]$$
$$= E_0\left[\sum_{t=0}^{T_n} \beta^t u_t\left(c_t^{\infty}\right) - \beta^t u_t\left(c_t^{\lim}\right)\right] \leq 0 \text{ for all } T_n. \tag{A11}$$

However, result (A11) contradicts to our assumption in (A9).

v). We conclude that for any subsequence $\{T_1, T_2, ...\} \subseteq \{0, 1, ...\}$, we have

$$E_0\left[\sum_{t=0}^{T_n} \beta^t \left\{u_t\left(c_t^{\infty}\right) - u_t(c_t^{\lim})\right\}\right] \leq 0 \text{ for all } T_n. \tag{A12}$$

However, under Assumptions 1 and 2, the optimal program $\left\{c_t^{\infty}, k_{t+1}^{\infty}\right\}_{t=0}^{\infty} \in \Im^{\infty}$ is unique, and hence, it must be that $\left\{c_t^{\infty}, k_{t+1}^{\infty}\right\}_{t=0}^{\infty}$ coincides with $\left\{c_t^{\lim}, k_{t+1}^{\lim}\right\}_{t=0}^{\infty}$ for all $t \geq 0$. ■

Appendix A6. Proof of the turnpike theorem

We now combine the results of Lemmas 1-3 together into a turnpike-style theorem to show the convergence of the optimal program of the T-period stationary economy to that of the infinite-horizon nonstationary economy. To be specific, Lemma 1 shows that the optimal program of the finite-horizon economy with a zero terminal condition $\left\{c_t^{T,0}, k_{t+1}^{T,0}\right\}_{t=0}^{T} \in \Im^{T,0}$ converges to the limit program $\left\{c_t^{\lim}, k_{t+1}^{\lim}\right\}_{t=0}^{\infty}$. Lemma 2 shows that the optimal program of the T-period stationary economy $\left\{c_t^{T,\kappa}, k_{t+1}^{T,\kappa}\right\}_{t=0}^{T}$ also converges to the same limit program $\left\{c_t^{\lim}, k_{t+1}^{\lim}\right\}_{t=0}^{\infty}$.

Finally, Lemma 3 shows that the limit program of the finite-horizon economies $\{c_t^{\lim}, k_{t+1}^{\lim}\}_{t=0}^{\infty}$ is optimal in the nonstationary infinite-horizon economy. Then, it must be the case that the limit optimal program of the T-period stationary economy $\left\{c_t^{T,\kappa}, k_{t+1}^{T,\kappa}\right\}_{t=0}^{T}$ is optimal in the infinite-horizon nonstationary economy. This argument is formalized below.

Proof of Theorem 2 (turnpike theorem). The proof follows by definition of limit and Lemmas 1-3. Let us fix a real number $\varepsilon > 0$ and a natural number τ such that $1 \leq \tau < \infty$ and consider a possible partial history $h_T = (\epsilon_0, ..., \epsilon_T)$.

i). Lemma 1 shows that $\left\{c_t^{T,0}, k_{t+1}^{T,0}\right\}_{t=0}^{T} \in \Im^{T,0}$ converges to a limit program $\{c_t^{\lim}, k_{t+1}^{\lim}\}_{t=0}^{\infty}$ as $T \to \infty$. Then, definition of limit implies that there exists $T_1(h_T) > 0$ such that $\left|k_{t+1}^{T,0} - k_{t+1}^{\lim}\right| < \frac{\varepsilon}{3}$ for $t = 0, ..., \tau$.

ii). Lemma 2 implies that the finite-horizon problem of the T-period stationary economy $\left\{c_t^{T,\kappa}, k_{t+1}^{T,\kappa}\right\}_{t=0}^{T}$ also converges to limit program $\{c_t^{\lim}, k_{t+1}^{\lim}\}_{t=0}^{\infty}$ as $T \to \infty$. Then, there exists $T_2(h_T) > 0$ such that $\left|k_{t+1}^{\lim} - k_{t+1}^{T,\kappa}\right| < \frac{\varepsilon}{3}$ for $t = 0, ..., \tau$.

iii). Lemma 3 implies the program $\left\{c_t^{T,0}, k_{t+1}^{T,0}\right\}_{t=0}^{T} \in \Im^{T,0}$ converges to the infinite-horizon optimal program $\{c_t^{\infty}, k_{t+1}^{\infty}\}_{t=0}^{\infty}$ as $T \to \infty$. Then, there exists $T_3(h_T) > 0$ such that $\left|k_{t+1}^{T,0} - k_{t+1}^{\infty}\right| < \frac{\varepsilon}{3}$ for $t = 0, ..., \tau$.

iv). Then, the triangular inequality implies

$$\begin{aligned}\left|k_{t+1}^{T,\kappa} - k_{t+1}^{\infty}\right| &= \left|k_{t+1}^{T,\kappa} - k_{t+1}^{\lim} + k_{t+1}^{\lim} - k_{t+1}^{T,0} + k_{t+1}^{T,0} - k_{t+1}^{\infty}\right| \\ &\leq \left|k_{t+1}^{T,\kappa} - k_{t+1}^{\lim}\right| + \left|k_{t+1}^{\lim} - k_{t+1}^{T,0}\right| + \left|k_{t+1}^{T,0} - k_{t+1}^{\infty}\right| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,\end{aligned}$$

for $T(h_T) \geq \max\{T_1(h_T), T_2(h_T), T_3(h_T)\}$.

v). Finally, consider all possible partial histories $\{h_T\}$ and define $T^*\left(\varepsilon, \tau, x_T^T\right) \equiv \max_{\{h_T\}} T(h_T)$. By construction, for any $T > T^*\left(\varepsilon, \tau, x_T^T\right)$, the inequality (16) holds. ■

Remark A1. Our proof of the turnpike theorem addresses a technical issue that does not arise in the literature that focuses on finite-horizon economies with a zero terminal condition; see, e.g., Majumdar and Zilcha (1987), Mitra and Nyarko (1991) and Joshi (1997). Their construction relies on the fact that the optimal program of the finite-horizon economy is always pointwise below the optimal program of the infinite-horizon economy, i.e., $k_{t+1}^{T,\kappa} \leq k_{t+1}^{\infty}$, for $t = 1, ..., \tau$, and it gives strictly higher expected utility up to T than does the infinite-horizon optimal program (because excess capital can be consumed at terminal period T). This argument does not directly applies to our T-period stationary economy: our finite-horizon program can be either below or above the infinite-horizon program depending on a specific T-period terminal condition. Our proof addresses this issue by constructing in Lemma 2 a separate limit program for the T-period stationary economy.

Appendix B. Implementation of EFP for growth model

In this section, we describe the implementation of the EFP method used to produce the numerical results in the main text.

Algorithm 1a (implementation):Extended function path (EFP) for the growth model.

The goal of EFP.

EFP is aimed at approximating a solution of a nonstationary model during the first τ periods, i.e., it finds approximating functions $\left(\widehat{K}_0, ..., \widehat{K}_\tau\right)$ such that $\widehat{K}_t \approx K_t$ for $t = 1, ...\tau$, where K_t and $\widehat{K}_t$ are a t-period true capital function and its parametric approximation, respectively.

Step 0. Initialization.

a. Choose time horizon $T \gg \tau$ for constructing T-period stationary economy.
b. Construct a deterministic path $\{z_t^*\}_{t=0}^T$ for exogenous state variable $\{z_t\}_{t=0}^T$ satisfying
$z_{t+1}^* = \varphi_t\left(z_t^*, E_t\left[\epsilon_{t+1}\right]\right)$ for $t = 0, .., T$.
c. Construct a deterministic path $\{k_t^*\}_{t=0}^T$ for endogenous state variable $\{k_t\}_{t=0}^T$ satisfying
$u_t'(c_t^*) = \beta u_t'(c_{t+1}^*)(1 - \delta + f_{t+1}'\left(k_{t+1}^*, z_{t+1}^*\right))$.
$c_t^* + k_{t+1}^* = (1 - \delta)\, k_t^* + f_t\left(k_t^*, z_t^*\right)$ for $t = 0, .., T$.
d. For $t = 0, ..., T$:
Construct a grid $\{(k_{m,t}, z_{m,t})\}_{m=1}^M$ centered at (k_t^*, z_t^*).
Choose integration nodes, $\epsilon_{j,t}$, and weights, $\omega_{j,t}$ for $j = 1, ..., J$.
Construct future shocks $z_{m,j,t}' = \varphi_t\left(z_{m,t}, \epsilon_{j,t}\right)$.
e. Write a t-period discretized system of the optimality conditions:

i). $u_t'(c_{m,t}) = \beta \sum_{j=1}^{J} \omega_{j,t}\left[u_t'(c_{m,j,t}')\left\{1 - \delta + f_{t+1}\left(k_{m,t}', z_{m,j,t}'\right)\right\}\right]$

ii). $c_{m,t} + k_{m,t}' = (1 - \delta)\, k_{m,t} + f_t\left(k_{m,t}, z_{m,t}\right)$

iii). $c_{m,j,t}' + k_{m,j,t}'' = (1 - \delta)\, k_{m,t}' + f_{t+1}\left(k_{m,t}', z_{m,j,t}'\right)$

iv). $k_{m,t}' = \widehat{K}_t\left(k_{m,t}, z_{m,t}\right)$ and $k_{m,j,t}'' = \widehat{K}_{t+1}\left(k_{m,t}', z_{m,j,t}'\right)$.

d. Assume that the model becomes stationary at T.

Step 1: Terminal condition.

Find $\widehat{K}_T = \widehat{K}_{T+1}$ that approximately solves the system i).-iv). on the grid for the T-period stationary economy $f_{T+1} = f_T$, $u_{T+1} = u_T$, $\varphi_{T+1} = \varphi_T$.

Step 2: Backward induction.

a. Construct the function path $\left(\widehat{K}_0, ..., \widehat{K}_{T-1}, \widehat{K}_T\right)$ that approximately solves the system i).-iv) for each $t = 0, ..., T$ and that matches the given terminal function $\widehat{K}_T$ constructed in Step 1.

Step 3: Turnpike property.

a. Simulate the process $\widehat{K}_0$ and use a subset of simulated points as initial conditions (k_0, z_0). For each initial condition, draw a history $h_\tau = (\epsilon_0, ..., \epsilon_\tau)$. Use the decision functions $\left(\widehat{K}_0, ..., \widehat{K}_\tau\right)$ to simulate the economy's trajectories $(k_0^T, ...k_\tau^T)$. Check that the trajectories converge to a unique limit $\lim_{T\to\infty}(k_0^T, ...k_\tau^T) = (k_0^*, ...k_\tau^*)$ by constructing $(K_0, ..., K_T)$ under different T and K_T.

The EFP solution:

Use $\left(\widehat{K}_0, ..., \widehat{K}_\tau\right)$ as an approximation to $(K_0, ..., K_\tau)$ and discard the remaining $T - \tau$ functions.

The EFP method is more expensive than conventional solution methods for stationary models because decision functions must be constructed not just once but for T periods. We implement EFP in the way that keeps its cost relatively low: First, to approximate decision functions, we use a version of the Smolyak (sparse) grid technique. Specifically, we use a version of the Smolyak method that combines a Smolyak grid with ordinary polynomials for approximating functions off the grid. This method is described in Maliar et al. (2011) who find it to be sufficiently accurate in the context of a similar growth model, namely, unit-free residuals in the model's equations do not exceed 0.01% on a stochastic simulation of 10,000 observations). For this version of the Smolyak method, the polynomial coefficients are overdetermined, for example, in a 2-dimensional case, we have 13 points in a second-level Smolyak grid, and we have only six coefficients in second-degree ordinary polynomial. Hence, we identify the coefficients using a least-squares regression; we use an SVD decomposition, to enhance numerical stability; see Judd et al. (2011) for a discussion of this and other numerically stable approximation methods. We do not construct the Smolyak grid within a hypercube normalized to $[-1,1]^2$, like do Smolyak methods that rely on Chebyshev polynomials used in, e.g., Krueger and Kubler (2004), Malin et al. (2011) and Judd et al. (2014). Instead, we construct a sequence of Smolyak grids around actual steady state and thus, the hypercube, in which the Smolyak grid is constructed, grows over time as shown in Figures 1 and 8.

Second, to approximate expectation functions, we use Gauss-Hermite quadrature rule with 10 integration nodes. However, a comparison analysis in Judd et al. (2011) shows that for models with smooth decision functions like ours, the number of integration nodes plays only a minor role in the properties of the solution, for example, the results will be the same up to six digits of precision if instead of ten integration nodes we use just two nodes or a simple linear monomial rule (a two-node Gauss-Hermite quadrature rule is equivalent to a linear monomial integration rule for the two-dimensional case). However, simulation-based Monte-Carlo-style integration methods produce very inaccurate approximations for integrals and are not considered in this paper; see Judd et al. (2011) for discussion.

Third, to solve for the coefficients of decision functions, we use a simple derivative-free fixed-point iteration method in line with Gauss-Jacobi iteration. Let us re-write the Euler equation i). constructed in the initialization step of the algorithm by pre-multiplying both sides by t-period capital

$$\widehat{k}'_{m,t} = \beta \sum_{j=1}^{J} \epsilon_{j,t} \left[\frac{u'_t(c'_{m,j,t})}{u'_t(c_{m,t})} \left\{ 1 - \delta + f_{t+1}\left(k'_{m,t} k^*_{t+1}, z'_{m,j,t} z^*_{t+1} \right) \right\} \right] k'_{m,t}. \tag{43}$$

We use different notation, $k'_{m,t}$ and $\widehat{k}'_{m,t}$, for t-period capital in the left and right side of (43), respectively, in order to describe our fixed-point iteration method. Namely, we substitute $k'_{m,t}$ in the right side of (43) and in the constraints ii). and iii). in the initialization step to compute $c_{m,t}$ and $c'_{m,j,t}$, respectively, and we obtain a new set of values of the capital function on the grid $\widehat{k}'_{m,t}$ in the left side. We iterate on these steps until convergence.

Our approximation functions $\widehat{K}_t$ are ordinary polynomial functions characterized by a time-dependent vector of parameters b_t, i.e., $\widehat{K}_t = \widehat{K}(\cdot; b_t)$. So, operationally, the iteration is performed not on the grid values $k'_{m,t}$ and $\widehat{k}'_{m,t}$ but on the coefficients of the approximation functions. The iteration procedure differs in Steps 1 and 2.

In Step 1, we construct a solution to T-period stationary economy. For iteration i, we fix some initial vector of coefficients b, compute $k'_{m,T+1} = \widehat{K}\left(k_{m,T}, z_{m,T}; b\right)$, find $c_{m,T}$ and $c'_{m,j,T}$ to satisfy constraints ii) and iii), respectively and find $\widehat{k}'_{m,T+1}$ from the Euler equation i). We run a regression of $\widehat{k}'_{m,T+1}$ on $\widehat{K}\left(k_{m,T}, z_{m,T}; \cdot\right)$ in order to re-estimate the coefficients $\widehat{b}$ and we compute the coefficients for iteration $i+1$ as a weighted average, i.e., $b^{(i+1)} = (1-\xi)\, b^{(i)} + \xi \widehat{b}^{(i)}$, where $\xi \in (0,1)$ is a damping parameter (typically, $\xi = 0.05$). We use partial updating instead of full updating $\xi = 1$ because fixed-point iteration can be numerically unstable and using partial updating enhances numerical stability; see Maliar et al. (2011). This kind of fixed-point iterations are used by numerical methods that solve for equilibrium in conventional stationary Markov economies; see e.g., Judd et al. (2011, 2014).

In Step 2, we iterate on the path for the polynomial coefficients using Gauss-Jacobi style iterations in line with Fair and Taylor (1983). Specifically, on iteration j, we take a path for the coefficients vectors $\left\{b_1^{(j)}, ..., b_T^{(j)}\right\}$, compute the corresponding path for capital quantities using $k'_{m,t} = \widehat{K}_t\left(k_{m,t}, z_{m,t}; b_t^{(j)}\right)$, and find a path for consumption quantities $c_{m,t}$ and $c'_{m,j,t}$ from constraints ii) and iii), respectively, for $t = 0, ..., T$. Substitute these quantities in the right side of a sequence of Euler equations for $t = 0, ..., T$ to obtain a new path for capital quantities in the left side of the Euler equation $\widehat{k}'_{m,t}$ for $t = 0, ..., T-1$. Run $T-1$ regressions of $\widehat{k}'_{m,t}$ on polynomial functional forms $\widehat{K}_t\left(k_{m,t}, z_{m,t}; b_t\right)$ for $t = 0, ..., T-1$ to construct a new path for the coefficients $\left\{\widehat{b}_0^{(j)}, ..., \widehat{b}_{T-1}^{(j)}\right\}$. Compute the path of the coefficients for iteration $j+1$ as a weighted average, i.e., $b_t^{(j+1)} = (1-\xi)\, b_t^{(j)} + \xi \widehat{b}_t^{(j)}$, $t = 0, ..., T-1$, where $\xi \in (0,1)$ is a damping parameter which we again typically set at $\xi = 0.05$. (Observe that this iteration procedure changes all the coefficients on the path except of the last one $b_T^{(j)} \equiv b$, which is a given terminal conditions that we computed in Step 1 from the T-period stationary economy).

In fact, the problem of constructing a path for function coefficients is similar to the problem of constructing a path for variables: in both cases, we need to solve a large system of nonlinear equations. The difference is that under EFP, the arguments of this system are not variables but parameters of the approximating functions. Instead of Gauss-Jacobi style iteration on path, we can use Gauss-Siedel fixed-point iteration (shooting), Newton-style solvers or any other technique that can solve a system of nonlinear equations; see Lipton et al. (1980), Atolia and Buffie (2009a,b), Heer and Maußner (2010), and Grüne et al. (2013) for examples of such techniques.

Let us now finally provide an additional illustration to the solution shown in Section 3.4. Specifically, in Figure 2, we plot a two-dimensional sequence of capital decision functions under fixed productivity level $z = 1$, while here we provide a three-dimensional plot of the same decision function for adding the productivity level. We again illustrate the capital functions for periods 1, 20 and 40, (i.e., $k_2 = K_1\left(k_1, z_1\right)$, $k_{21} = K_{20}\left(k_{20}, z_{20}\right)$ and $k_{41} = K_{40}\left(k_{40}, z_{40}\right)$) which we approximate using Smolyak (sparse) grids. In Step 1 of the algorithm, we construct the capital function K_{40} by assuming that the economy becomes stationary in period $T = 40$; and in Step 2, we construct a path of the capital functions that $(K_1, ...K_{39})$ that matches the corresponding terminal function K_{40}. The Smolyak grids are shown by stars in the horizontal $k_t \times z_t$ plane. The domain for capital (on which Smolyak grids are constructed) and the range of the constructed capital function grow at the rate of labor-augmenting technological progress.

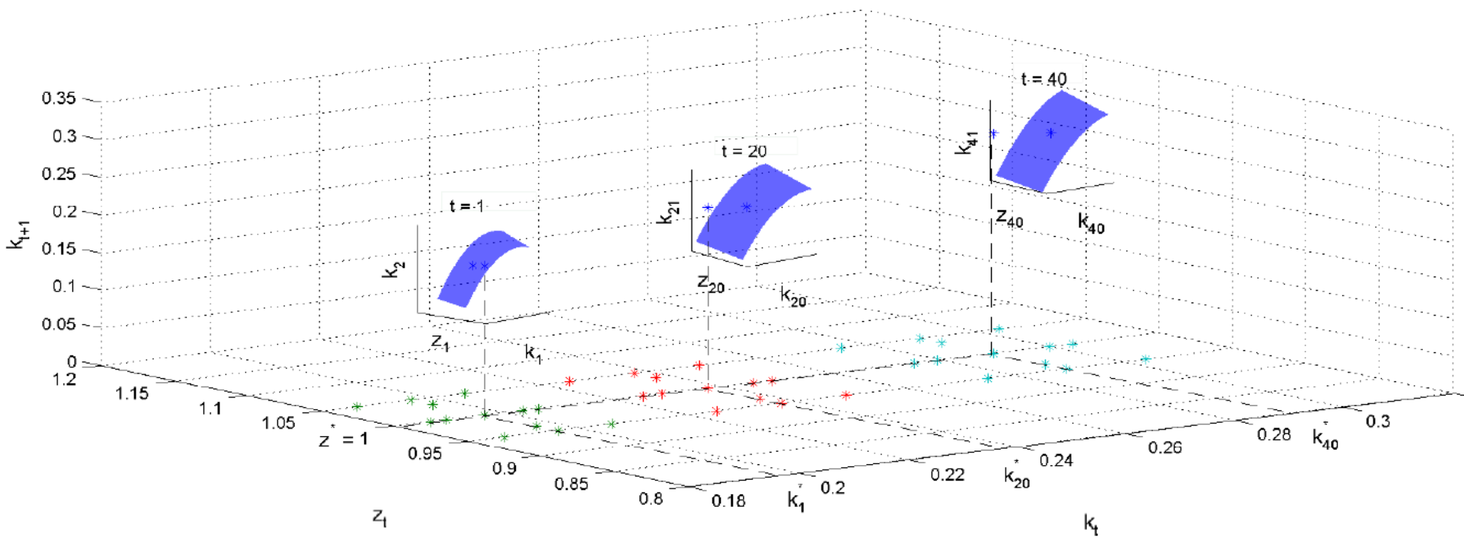

Figure 8. Function path, produced by EFP, for a growth model with technological progress.

Appendix C. Path-solving methods for nonstationary models

We first describe the shooting method of Lipton et al. (1980) for a nonstationary deterministic economy, and we then elaborate the extended path (EP) of Fair and Taylor (1983) for a nonstationary economy with uncertainty.

Shooting methods. To illustrate the class of shooting methods, let us substitute c_t and c_{t+1} from (37) into the Euler equation of (36)–(38) to obtain a second-order difference equation,

$$u_t'((1-\delta)\,k_t + f_t\,(k_t, z_t) - k_{t+1}) \\ = \beta E_t \left[u_{t+1}'((1-\delta)\,k_{t+1} + f_{t+1}\,(k_{t+1}, z_{t+1}) - k_{t+2})(1-\delta + f_{t+1}'\,(k_{t+1}, z_{t+1}))\right]. \quad (44)$$

Initial condition (k_0, z_0) is given. Let us abstract from uncertainty by assuming that $z_t = 1$ for all t, choose a sufficiently large T and fix some terminal condition k_{T+1} (typically, the literature assumes that the economy arrives in the steady state $k_{T+1} = k^*$).[14] To approximate the optimal path, we must solve numerically a system of T nonlinear equations (44) with respect to T unknowns $\{k_1, ..., k_T\}$. It is possible to solve the system (44) by using a Newton-style or any other numerical solver. However, a more efficient alternative could be numerical methods that exploit the recursive structure of the system (44) such as shooting methods (Gauss-Siedel iteration). There are two types of shooting methods: a forward shooting and a backward shooting. A typical forward shooting method expresses k_{t+2} in terms of k_t and k_{t+1} using (44)

[14] The turnpike theorem implies that in initial τ periods, the optimal path is insensitive to a specific terminal condition used if $\tau \ll T$.

and constructs a forward path $(k_1, ..., k_{T+1})$; it iterates on k_1 until the path reaches a given terminal condition $k_{T+1} = k^*$. In turn, a typical reverse shooting method expresses k_t in terms of k_{t+1} and k_{t+2} and constructs a backward path $\{k_T, ..., k_0\}$; it iterates on k_T until the path reaches a given initial condition k_0. A shortcoming of shooting methods is that they tend to produce explosive paths, in particular, forward shooting methods; see Atolia and Buffie (2009 a, b) for a careful discussion and possible treatments of this problem.

Fair and Taylor (1984) method. The EP method of Fair and Taylor (1983) allows us to solve nonstationary economic models with uncertainty by approximating expectation functions under the assumption of certainty equivalence. To see how this method works, consider the system (44) with uncertainty and as an example, assume that z_{t+1} follows a possibly nonstationary Markov process $\ln(z_{t+1}) = \rho_t \ln(z_t) + \sigma_t \epsilon_{t+1}$, where the sequences $(\rho_0, \rho_1, ...)$ and $(\sigma_0, \sigma_1, ...)$ are deterministically given at $t = 0$ and $\epsilon_{t+1} \sim \mathcal{N}(0, 1)$. Again, let us choose a sufficiently large T and fix some terminal condition such as $k_{T+1} = k^*$, so that the turnpike argument applies. Fair and Taylor (1983) propose to construct a solution path to (44) by setting all future innovations to their expected values, $\epsilon_1 = \epsilon_2 = ... = 0$. This produces a path on which technology evolves as $\ln(z_{t+1}) = \rho_t \ln(z_t)$ gradually converging to $z^* = 1$ and the models's variables gradually converge to the steady state. Note that only the first entry k_1 of the constructed path $(k_1, ..., k_T)$ is meaningful; the remaining entries $(k_2, ..., k_T)$ are obtained under a supplementary assumption of zero future innovations and they are only needed to accurately construct k_1. Thus, k_1 is stored and the rest of the sequence is discarded. By applying the same procedure to next state (k_1, z_1), we produce k_2, and so on until the path of desired length τ is constructed.

However, certainty equivalence approximation of Fair and Taylor (1983) has its limitations. It is exact for linear and linearized models, and it can be sufficiently accurate for models that are close to linear; see Cagnon and Taylor (1990), and Love (2010). However, it becomes highly inaccurate when either volatility and/or the degrees of nonlinearity increase; see our accuracy evaluations in Section 4.

Another novelty of the EP method relative to shooting methods is that it iterates on the economy's path at once using Gauss-Jacobi iteration. This type of iteration is more stable than Gauss-Siedel and allows us to avoid explosive behavior. To be specific, it guesses the economy's path $(k_1, ..., k_{T+1})$, substitute the quantities for $t = 1, ...T + 1$ it in the right side of T Euler equations (44), respectively, and obtains a new path $(k_0, ..., k_T)$ in the left side of (44); and it iterates on the path until the convergence is achieved. Finally, Fair and Taylor (1983) propose a simple procedure for determining T that insures that a specific terminal condition used does not affect the quality of approximation, namely, they suggested to increase T (i.e., extend the path) until the solution in the initial period(s) becomes insensitive to further increases in T.

We now elaborate the description of the version of Fair and Taylor's (1983) method used to produce the results in the main text. We use a slightly different representation of the optimality conditions of the model (36)–(38) (we assume $\delta = 1$ and $u(c) = \ln(c)$ for expository convenience). The Euler equation and budget constraint, respectively, are:

$$\frac{1}{c_t} = \beta E_t \left[\frac{1}{c_{t+1}} (1 - \delta + z_{t+1} f'(k_{t+1})) \right],$$
$$c_t + k_{t+1} = (1 - \delta) k_t + z_t f(k_t).$$

We combine the above two conditions to get

$$k_{t+1} = z_t f\left(k_t\right) - \left[E_t\left(\frac{\beta z_{t+1} f'(k_{t+1}))}{z_{t+1} f\left(k_{t+1}\right) - k_{t+2}}\right)\right]^{-1} \approx$$

$$z_t f\left(k_t\right) - \frac{z^e_{t+1} f\left(k_{t+1}\right) - k_{t+2}}{\beta z^e_{t+1} f'(k_{t+1}))}, \qquad (45)$$

where the path for $z^e_{t+1}$is constructed under the certainty equivalence assumption that $\epsilon_{t+1} = 0$ for all $t \geq 0$. Under the conventional AR(1) process for productivity levels (3), this means that $\ln z^e_{t+1} = \rho \ln z^e_t$ for all $t \geq 0$, or equivalently $z^e_{t+1} = (z^e_t)^\rho$, where $z^e_0 = z_0$. To solve for the path of variables, we use derivative-free iteration in line with Gauss-Jacobi method as in Fair and Taylor (1983):

Algorithm 2. Extended path (EP) framework by Fair and Taylor (1983).

The goal of EP framework of Fair and Taylor (1983).
EFP is aimed at approximating a path for variables satisfying the model's equations during the first τ periods, i.e., it finds $\widehat{k}_0, ..., \widehat{k}_\tau$ such that $\left\Vert k_t - \widehat{k}_t \right\Vert < \varepsilon$ for $t = 1, ...\tau$, where $\varepsilon > 0$ is target accuracy, $\Vert\cdot\Vert$ is an absolute value, and k_t and $\widehat{k}_t$ are the t-period true capital stocks and their approximation, respectively.
Step 0: Initialization.
a. Fix $t = 0$ period state (k_0, z_0). b. Choose time horizon $T \gg \tau$ and terminal condition $\widehat{k}_{T+1}$. c. Construct and fix $\left\{z^e_{t+1}\right\}_{t=0,...,T}$ such that $z^e_{t+1} = (z^e_t)^\rho$ for all t, where $z^e_0 = z_0$. d. Guess an equilibrium path $\left\{\widehat{k}^{(1)}_t\right\}_{t=1,...,T'}$ for iteration $j = 1$. e. Write a t-period system of the optimality conditions in the form: $\widehat{k}_{t+1} = z^e_t f\left(\widehat{k}_t\right) - \frac{z^e_{t+1} f\left(\widehat{k}_{t+1}\right) - \widehat{k}_{t+2}}{\beta z^e_{t+1} f'(\widehat{k}_{t+1}))}$, where $\widehat{k}_0 = k_0$.
Step 1: Solving for a path using Gauss-Jacobi method.
a. Substitute a path $\left\{\widehat{k}^{(j)}_t\right\}_{t=1,...,T'}$ into the right side of (45) to find $\widehat{k}^{(j+1)}_{t+1} = z^e_t f\left(\widehat{k}^{(j)}_t\right) - \frac{z^e_{t+1} f\left(\widehat{k}^{(j)}_{t+1}\right) - \widehat{k}^{(j)}_{t+2}}{\beta z^e_{t+1} f'(\widehat{k}^{(j)}_{t+1}))}, \qquad t = 1, ..., T$ b. End iteration if the convergence is achieved $\left\vert \widehat{k}^{(j+1)}_{t+1} - \widehat{k}^{(j)}_{t+1} \right\vert <$ tolerance level. Otherwise, increase j by 1 and repeat Step 1.
The EP solution:
Use the first entry $\widehat{k}_1$ of the constructed path $\widehat{k}_1, ..., \widehat{k}_T$ as an approximation to the true solution k_1 n period $t = 0$ and discard the remaining $k_2, ..., k_T$ values.

In terms of our notations, Fair and Taylor (1983) use $\tau = 1$, i.e., they keep only the first element $\widehat{k}_1$ from the constructed path $\left(\widehat{k}_1, ..., \widehat{k}_T\right)$ and disregard the rest of the path; then, they draw a next period shock z_1 and solve for a new path $\left(\widehat{k}_1, ..., \widehat{k}_{T+1}\right)$ starting from $\widehat{k}_1$ and ending in a given $\widehat{k}_{T+1}$ and store $\widehat{k}_2$, again disregarding the rest of the path; and they advance forward until the path of the given length τ is constructed. T is chosen so that its further extensions do not affect the solution in the initial period of the path. For instance, to find a solution $\widehat{k}_1$, Fair and Taylor (1983) solve the model several times under $T+1$, $T+2$, $T+3, ...$ and check that $\widehat{k}_1$ remains the same (up to a given degree of precision).

As is typical for fixed-point-iteration style methods, Gauss-Jacobi iteration may fail to converge. To deal with this issue, Fair and Taylor (1983) use damping, namely, they update the path over iteration only by a small amount $k_{t+1}^{(j+1)} = \xi k_{t+1}^{(j+1)} + (1-\xi)\, k_{t+1}^{(j)}$ where $\xi \in (0,1)$ is a small number close to zero (e.g., 0.01).

Steps 1a and *1b* of Fair and Taylor's (1983) method are called Type I and Type II iterations and are analogous to *Step 2* of the EFP method when the sequence of the decision functions is constructed. The extension of path is called Type III iteration and gives the name to Fair and Taylor (1983) method.

In our examples, we implement Fair and Taylor's (1983) method using a conventional Newton style numerical solver instead of Gauss-Jacobi iteration; a similar implementation is used in Heer and Maußner (2010). The cost of Fair and Taylor's (1983) method can depend considerably on a specific solver used and can be very high (as we need to solve a system of equations with hundreds of unknowns numerically). In our simple examples, a Newton-style solver was sufficiently fast and reliable. In more complicated models, we are typically unable to derive closed-form laws of motion for the state variables, and derivative-free fixed-point iteration advocated in Fair and Taylor (1983) can be a better alternative.

Appendix D. Solving the test model using the associated stationary model

We consider model (36)–(38) parameterized by (13) and (12). We first convert the nonstationary with labor-augmenting technological progress into a stationary model using the standard change of variables $\widehat{c}_t = c_t/A_t$ and $\widehat{k}_t = k_t/A_t$. This leads us to the following model

$$\max_{\left\{\widehat{k}_{t+1},\widehat{c}_t\right\}_{t=0,...,\infty}} E_0 \sum_{t=0}^{\infty} (\beta^*)^t \frac{\widehat{c}_t^{1-\eta}}{1-\eta} \tag{46}$$

$$\text{s.t. } \widehat{c}_t + \gamma_A \widehat{k}_{t+1} = (1-\delta)\, \widehat{k}_t + z_t \widehat{k}_t^{\alpha}, \tag{47}$$

$$\ln z_{t+1} = \rho_t \ln z_t + \sigma_t \epsilon_{t+1}, \qquad \epsilon_{t+1} \sim \mathcal{N}(0,1), \tag{48}$$

where $\beta^* \equiv \beta \gamma_A^{1-\eta}$. We solve this stationary model by using the same version of the Smolyak method that is used within EFP to find a solution to T-period stationary economy.

After a solution to the stationary model (46)–(48) is constructed, a solution for nonstationary variables can be recovered by using an inverse transformation $c_t = \widehat{c}_t A_t$ and $k_t = \widehat{k}_t A_t$.

For the sake of our comparison, we also need to recover the path of nonstationary decision functions in terms of their parameters. Let us show how this can be done under polynomial approximation of decision functions. Let us assume that a capital policy function of the stationary model is approximated by complete polynomial of degree L, namely, $\hat{k}_{t+1} =$

$\sum_{l=0}^{L}\sum_{m=0}^{l} b_{m+\frac{(l-1)(l+2)}{2}+1}\hat{k}_t^m z_t^{l-m}$, where b_i is a polynomial coefficient, $i = 0, ..., L+\frac{(L-1)(L+2)}{2}+1$. Given that the stationary and nonstationary solutions are related by $\hat{k}_{t+1} = k_{t+1}/\left(A_0\gamma_A^{t+1}\right)$, we have

$$k_{t+1} = A_0\gamma_A^{t+1}\hat{k}_{t+1} = A_0\gamma_A^{t+1}\sum_{l=0}^{L}\sum_{m=0}^{l} b_{m+\frac{(l-1)(l+2)}{2}+1}\hat{k}_t^m z_t^{l-m} =$$

$$A_0\sum_{l=0}^{L}\sum_{m=0}^{l} {\gamma_A}^{1-(m-1)t} b_{m+\frac{(l-1)(l+2)}{2}+1} k_t^m z_t^{l-m}. \quad (49)$$

For example, for first-degree polynomial $L = 1$, we construct the coefficients vector of the nonstationary model by premultiplying the coefficient vector $b \equiv (b_0, b_1, b_2)$ of the stationary model by a vector $\left(A_0\gamma_A^{t+1}, A_0\gamma_A, A_0\gamma_A^{t+1}\right)^\top$, which yields $b_{t+1} \equiv \left(b_0 A_0\gamma_A^{t+1}, b_1 A_0\gamma_A, b_2 A_0\gamma_A^{t+1}\right)$, $t = 0, ..., T$, where T is time horizon (length of simulation in the solution procedure). Note that a similar relation will hold even if the growth rate γ_A is time variable.

Appendix E. Sensitivity results for the model with labor-augmenting technological progress

In this appendix, we provide sensitivity results for the model with labor-augmenting technological progress. Table 2 contains the results on accuracy and cost of the version of the EFP method studied in Section 5. We use $\tau = 200$ and $T = 400$ and consider several alternative parameterizations for $\{\eta, \sigma_\epsilon, \gamma_A\}$.

Figure E.1 plots a maximum unit-free absolute difference between the exact solution for capital and the solution delivered by the EFP at $\tau = 100$. The difference between the solutions is computed across 1,000 simulations. We use $T = \{200, 300, 400, 500\}$, $\eta = \{1/3, 1, 3\}$ and decision rules produced by the T-period stationary economy and zero capital assumption as terminal conditions.

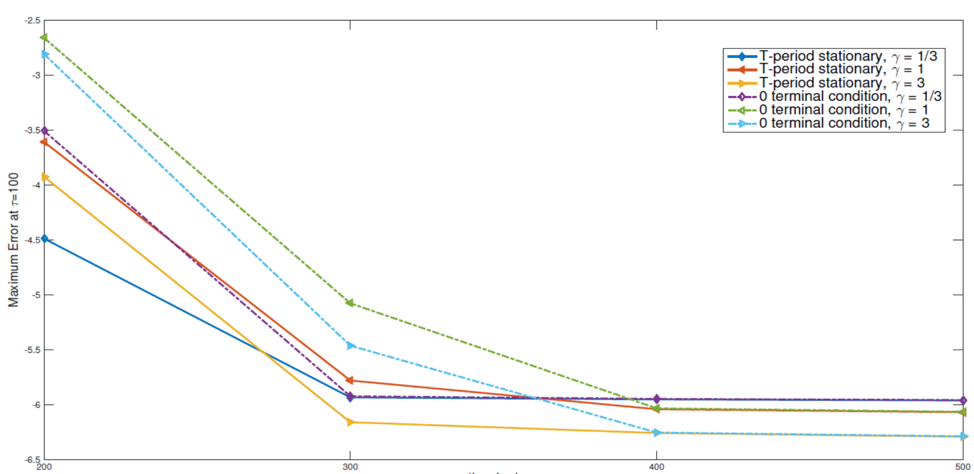

Figure 9. Sensitivity analysis for the EFP method.

Table 2: Sensitivity analysis for the EFP method.

Parameters	Model 1	Model 2	Model 3	Model 4	Model 5	Models 6	Model 7
η	5	5	5	5	0.1	1	10
σ_ϵ	0.03	0.03	0.03	0.01	0.01	0.01	0.01
γ_A	1.01	1.00	1.05	1.01	1.01	1.01	1.01
Mean errors across t periods in $\log_{10}$ units							
$t \in [0, 50]$	-7.01	-6.67	-7.34	-7.03	-7.03	-6.61	-7.30
$t \in [0, 100]$	-6.82	-6.44	-7.25	-6.84	6.92	-6.48	-7.08
$t \in [0, 150]$	-6.73	-6.33	-7.22	-6.76	-6.89	-6.43	-6.98
$t \in [0, 175]$	-6.70	-6.29	-7.22	-6.74	-6.87	-6.41	-6.95
$t \in [0, 200]$	-6.68	-6.26	-7.21	-6.72	-6.87	-6.37	-6.93
Maximum errors across t periods in $\log_{10}$ units							
$t \in [0, 50]$	-6.42	-6.31	-7.13	-6.66	-6.08	-6.24	-6.81
$t \in [0, 100]$	-5.99	-6.12	-7.05	-6.54	-5.97	-6.18	-6.36
$t \in [0, 150]$	-5.98	-6.04	-7.05	-6.52	-5.97	-6.18	-6.35
$t \in [0, 175]$	-5.98	-6.01	-7.05	-6.52	-5.97	-6.13	-6.33
$t \in [0, 200]$	-5.92	-5.99	-7.05	-6.51	-5.96	-5.88	-6.24
Running time, in seconds							
Solution	225.9	150.0	193.0	216.98	836.5	300.7	245.9
Simulation	5.6	5.7	5.8	5.66	5.6	5.6	5.7
Total	231.6	155.7	198.8	222.64	842.1	306.3	251.6

Notes: "Mean errors" and "Maximum errors" are, respectively, mean and maximum unit-free absolute difference between the exact solution for capital and the solution delivered by EFP under the parameterization in the column. The difference between the solutions is computed across 100 simulations. The time horizon is T=400, and the terminal condition is constructed by using the T-period stationary economy in all experiments.

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