DISCUSSION PAPER SERIES

DP13383
(v. 3)

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MONETARY ECONOMICS AND FLUCTUATIONS

ISSN 0265-8003

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Discussion Paper DP13383
First Published 10 December 2018
This Revision 14 July 2020

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JEL Classification: C61, C63, C68, E31, E52

Keywords: forward guidance, New Keynesian Model

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Acknowledgements
This paper was initially titled "Forward Guidance: Is It Useful Away from the Lower Bound?"

Taylor Rules and Forward Guidance: A Rule is not a Path*

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June 23, 2020

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*This paper was initially titled "Forward Guidance: Is It Useful Away from the Lower Bound?" NBER working paper 26053.

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1 Introduction

Forward guidance (FG) refers to central banks' announcements regarding future federal fund rates. A possibility of using FG as a policy instrument was suggested in a pioneering work of Eggerstson and Woodford (2003) who argued that future commitments can lead to an increase in current output. During the Great Recession and its aftermath, FG was widely used by central banks as an unconventional monetary policy tool when nominal interest rates were constrained at their effective lower bounds (ELB). For example, in August 2011, the Fed signalled an intention to maintain nominal interest rates at zero at least until the middle of 2013. Numerous papers study the impact of forward guidance on the economy, including Werning (2015), Cochrane (2017a), Del Negro et al. (2015), Carlstrom et al. (2015), and McKay et al. (2016). In particular, this literature documented the so-called *forward guidance puzzle* – a counterintuitive implication of the new Keynesian model that the central bank's announcements about future interest rates have an immediate and unrealistically large impact on the economy. The effectiveness of forward guidance as a policy tool still remains a subject of debate.

In the paper, we contribute to this debate by characterizing the impact of forward guidance announcements in a stylized new Keynesian model. Using novel closed-form solutions, we show that the effectiveness of guidance on the economy depends critically on how active the monetary policy. Specifically, if a monetary authority uses a sufficiently active Taylor-style rule which facilitates the economy stabilization, the effect of distant shocks on the economy is negligible. However, if the monetary authority uses a relatively passive rule, we obtain a backward explosion, leading to the FG puzzle. The puzzle is accentuated if the monetary authority follows a pre-specified interest rate path – an extremely passive policy which contemplates no reaction to the ongoing economic conditions. Werning (2015) and Cochrane (2017a) pointed out that the policy of choosing the interest rate path directly is a coherent way of modeling monetary policy at the ELB where active Taylor-style rules are infeasible.

We delineate the parameter space into four regions characterized by different characteristic roots (or eigenvalues). The key determinants of the constructed regions are the coefficients of the Taylor rule – the responsiveness of the interest rate to output, inflation, and expected inflation. We show that the characteristic roots capture the properties of the model's dynamics better than the underlying model parameters. Specifically, the economy is backward stable as long as both roots are unstable (either real or complex), however, it explodes backward if one of the two roots becomes stable. Our theorems give a simple recipe for testing the effectiveness of FG: the size of the smallest root is a single sufficient statistics to determine whether or not the model generates a FG puzzle.

As an illustration, consider the Taylor rule that contains just a feedback to inflation.[1] First, if the inflation coefficient in the Taylor rule is smaller than one, the monetary policy is relatively irresponsive to shocks, leading to a stable root. In the limit, this case includes a zero inflation coefficient that corresponds to the policy of choosing the interest-rate path directly. Werning (2015) and Cochrane (2017a) demonstrates that such policy leads to a multiplicity of equilibrium and a pronounced backward explosion. Our theorem shows that these implications are not limited to the policy of the interest-rate path but that they are a generic property of the model in which the monetary policy rule is not sufficiently responsive to insure the equilibrium uniqueness and stabilization.

Second, when the inflation coefficient in the Taylor rule approaches one, the smallest root also approaches one, which makes the equilibrium unique. A version of this special but conceptually important case is studied in McKay et al. (2016), and it also leads to a FG puzzle, although less pronounced. Our theorem implies that in such a borderline case, the Taylor rule is still insufficiently active to ensure economy stabilization, so that the backward explosion persists as well.

Finally, by increasing the inflation coefficient beyond one, we arrive to the original Taylor (1993) rule in which the Taylor Principle that the response to inflation is greater than one holds. Our theorem shows that the smallest root is now unstable; this eliminates backward explosion, and the FG puzzle disappears.

[1] More general Taylor rules, considered in this paper, include a feedback to inflation, expected inflation and the output gap, and the responsiveness of monetary policy depends on all the feedback coefficients.

The Taylor principle provides a sufficient economy stabilization and leads to a unique equilibrium with sensible predictions: (i) the effect of FG is small if policy announcements refer to a distant future; (ii) such effect decreases with the horizon of a policy announcement (i.e., the further away in the future is anticipated interest-rate shock, the smaller is the effect of this shock on today's economy); (iii) the effect of FG on output depends on specific timing and nature of the policy announcement, in particular, FG can have a detrimental effect on output, at least during some periods, or it can lead to cyclical fluctuations of a decreasing amplitude.

Our findings are robust. We show that similar regularities hold under more general Taylor rules that include a feedback to inflation, expected inflation and the output gap. Also, our numerical analysis shows that they continue to hold in a fully non-linear version of the studied new Keynesian model with uncertainty, as well as in a version of the model augmented to include capital. Finally, our findings are robust to the solution methods used, specifically, both Fair and Taylor's (1983) extended path method and Maliar et al.'s (2020) extended function path method lead to similar results.

The previous FG literature offers a variety of ways of resolving the FG puzzle. In particular, Del Negro et al. (2015) constructed a perpetual-youth version of Smets and Wouters (2007) model in which the presence of cohorts results in heavier discounting of the future; McKay et al. (2016) introduced idiosyncratic household risk and borrowing constraints; Husted et al. (2017) modified their model to allow for policy uncertainty; Kaplan et al (2017) introduced heterogeneous agents; Gabaix (2017) assumed that agents are not fully rational and do not have perfect foresight of the future; Campbell et al. (2019) introduced imperfect communication that limits the Fed's ability to affect expectations at long horizons; etc.[2]

Unlike the above literature, we do not attempt to modify the baseline new Keynesian model to resolve the FG puzzle. Instead, we argue that the FG puzzle is a consequence of excessively passive monetary policies. In particular, in the example of Del Negro et al. (2015), the FG puzzle occurs because prices are fixed. In turn, McKay et al. (2016) assume that the monetary authority sets the nominal interest rate to completely accommodate the anticipated inflation. A pre-specified interest rate path studied in Werning (2015) and Cochrane (2017a) is an even more extreme policy in which the monetary authority commits not to react to changes in economic conditions no matter how the economy evolves. In the absence of active stabilization policy, future shocks are propagated backward without discounting or even amplified.

Nevertheless, such excessively passive policies are suboptimal because they do not stabilize inflation, while the optimal policy implies a zero inflation target. Inflation stabilization is also the key prescription of the Taylor rule used by actual central banks and it is in line with the optimal monetary policy.[3] With active stabilization, the agents will discount the impact of future shocks by anticipating future stabilization policies and this is precisely what prevents the backward explosion from happening. We conclude that it is not the ELB or the new Keynesian model per se that produced the FG puzzle but the assumption of unrealistically passive policy rules.

The rest of the paper is as follows: In Section 2, we derive closed-form solutions in the stylized new Keynesian model. In Section 3, we analyze the policy of choosing a pre-specified interest rate path. In Section 4, we study the FG puzzle in the economy with a unit root. In Section 5, we show that empirically relevant Taylor rules do not lead to backward explosion. In Section 5, we discuss the effectiveness of FG as a policy tool. Finally, in Section 6, we conclude.

2 Stylized new Keynesian model

In this section, we formulate a standard three-equation linear new Keynesian model, express it as a second-order difference equation, derive a closed-form solution and characterize some of its properties.

[2] Other papers that study the effectiveness of FG include Levin et al.(2010), Werning (2012), Den Haan (2013), Carlstrom et al. (2015), Chung (2015), Bundick and Smith (2016), Keen et al. (2016), Galí (2017), Walsh (2017), Hagedorn et al. (2018).

[3] Woodford (2001) shows that the optimal Ramsey rule closely resembles the empirically relevant Taylor rule with both inflation and output gap and with standard calibration of its coefficients.

2.1 The model

We consider the standard linearized new Keynesian model that consists of, respectively, an IS equation and Phillips curve, expressed in deviations from the steady state

$$x_t = E_t[x_{t+1}] - \sigma(i_t - i_t^n - E_t[\pi_{t+1}]), \tag{1}$$

$$\pi_t = \beta E_t[\pi_{t+1}] + \kappa(x_t + g_t), \tag{2}$$

where x_t is an output gap; π_t is inflation; i_t is a nominal interest rate; i_t^n is a natural rate of interest; g_t is a disturbance; κ is a slope of the Phillips curve; σ is an intertemporal elasticity of substitution.[4] The Phillips-curve shifter g_t can be interpreted as a direct marginal cost increase due to, for example, capital destruction, technical regress, government spending (up to a scaling factor)

For the third equation, we assume that the nominal interest rate i_t is determined by a stylized Taylor rule

$$i_t = i_t^* + \phi_\pi \pi_t + \phi_{E\pi} E_t[\pi_{t+1}] + \phi_y x_t + \varepsilon_t, \tag{3}$$

where $\{i_t^*\}$ is a desired interest rate path; $\phi_\pi \geq 0$, $\phi_{E\pi} \geq 0$ and $\phi_y \geq 0$ are constant coefficients; and ε_t is a disturbance that may include both anticipated and unanticipated shocks. The rule studied in Taylor (1993) corresponds to $\phi_\pi = 1.5$, $\phi_y = 0.5$, $\phi_{E\pi} = 0$ and $i_t^* = 1$.[5]

2.2 Characteristic roots

We next derive key regions to the parameters and eigenvalues of this system. It will be convenient to re-write the model (1)–(3) as a second-order difference equation. We substitute i_t from (3) into (1), use (2) to express x_t and x_{t+1} and substitute them into (1) to obtain

$$E_t[\pi_{t+2}] + bE_t[\pi_{t+1}] + c\pi_t = -z_t, \tag{4}$$

where $b \equiv -1 - \frac{1}{\beta} - \sigma\phi_y - \frac{\sigma\kappa(1-\phi_{E\pi})}{\beta}$; $c \equiv \frac{(1+\sigma\phi_y)}{\beta} + \frac{\sigma\kappa\phi_\pi}{\beta}$; z_t includes all exogenous variables, g_t, i_t^*, ε_t, i_t^n,

$$z_t \equiv \frac{\kappa}{\beta}\left[g_{t+1} - g_t\left(1 + \sigma\phi_y\right) + \sigma\left(i_t^* + \varepsilon_t - i_t^n\right)\right]. \tag{5}$$

Let us first construct a solution for the economy with perfect foresight by eliminating the expectation operator in (4) and later, we will show how to generalize the solution to the case of uncertainty; see Appendix B. Below, we establish some properties a homogenous equation $\pi_{t+2} + b\pi_{t+1} + c\pi_t = 0$ that corresponds to (4).

Theorem 1. *The roots* $m_{1,2} = \frac{-b\pm\sqrt{b^2-4c}}{2}$ *to characteristic equation* $m^2 + bm + c = 0$ *satisfy*

Case	Type of solution	Types of roots: unstable/ stable	distinct/ repeated	real/ complex	Restrictions on roots
i)	indeterminate	1 unstable, 1 stable	distinct	real	either $\lvert m_1\rvert > 1, \lvert m_2\rvert < 1$ or $\lvert m_1\rvert < 1, \lvert m_2\rvert > 1$
ii)	unique	2 unstable	distinct	real	$\lvert m_1\rvert \geq 1,\ \lvert m_2\rvert \geq 1$
iii)	unique	2 unstable	repeated	real	$m_1 = m_2 = m$ with $\lvert m\rvert > 1$
iv)	unique	2 unstable	distinct	complex	$m_{1,2} = \mu \pm \eta\iota$ with $r \equiv \sqrt{\mu^2+\eta^2} > 1$

[4] In Appendix D, we describe a fully nonlinear model whose linearized version corresponds to the model (1), (2) under some further restrictions. In particular, the slope of the Phillips curve is $\kappa = \frac{(1-\beta\theta)(1-\theta)}{\theta}(1+\vartheta)$, where β is a discount factor; θ is a share of not reoptimizing firms; ϑ is a parameter of the utility function $u(C_t, L_t) = \frac{C_t^{1-1/\sigma}-1}{1-1/\sigma} - \frac{L_t^{1+\vartheta}-1}{1+\vartheta}$; and $\sigma \to 1$.

[5] Our analysis abstracts from the issues of commitment, discretion and time inconsistency. These issues are studied, for example, in Walsh (2017).

Cases i)–iv) arise under the following restrictions on the Taylor rule's parameter $\phi_{E\pi}$*:*
i) $\phi_{E\pi} < \phi_{E\pi}^1$ *and* $\phi_{E\pi} > \phi_{E\pi}^4$*;*
ii) $\phi_{E\pi}^1 \leq \phi_{E\pi} < \phi_{E\pi}^2$ *and* $\phi_{E\pi}^3 \leq \phi_{E\pi} < \phi_{E\pi}^4$*;*
iii) $\phi_{E\pi} = \phi_{E\pi}^2$ *and* $\phi_{E\pi} = \phi_{E\pi}^3$*;*
iv) $\phi_{E\pi}^2 < \phi_{E\pi} < \phi_{E\pi}^3$*.*
Here, $\phi_{E\pi}^1 = 1-\phi_\pi-\frac{(1-\beta)\phi_y}{\kappa}$, $\phi_{E\pi}^2 = \phi_{E\pi}^1+\frac{1}{\sigma\kappa}\left(\sqrt{1+\phi_\pi\sigma\kappa+\sigma\phi_y}-\sqrt{\beta}\right)^2$, $\phi_{E\pi}^3 = \phi_{E\pi}^1+\frac{1}{\sigma\kappa}\left(\sqrt{1+\phi_\pi\sigma\kappa+\sigma\phi_y}+\sqrt{\beta}\right)^2$, $\phi_{E\pi}^4 = \phi_{E\pi}^1 + \frac{2}{\sigma\kappa}\left(1+\sigma\phi_y+\sigma\kappa\phi_\pi+\beta\right)$.

Proof. See Appendix A. ■

Thus, if the roots are real and distinct, there are two possibilities: either one root is stable and the other root is unstable (case i) or both roots are unstable (case ii). If the roots are either real and repeated (case iii) or complex (case iv), they are always unstable. Thus, it is never the case that both roots are stable in the considered area of the parameter space.

We have further results for the economy in which the rule (3) contains only actual but not future inflation.

Corollary 1. *Assume* $\phi_{E\pi} = 0$*. Then, cases i)–iv) specified in Theorem 1 arise under the following restrictions on the Taylor rule's parameter* ϕ_π*:*
i) $\phi_\pi < \phi_\pi^1$*;*
ii) $\phi_\pi^1 \leq \phi_\pi < \phi_\pi^2$*;*
iii) $\phi_\pi = \phi_\pi^2$*;*
iv) $\phi_\pi > \phi_\pi^2$*.*
Here, $\phi_\pi^1 = 1 - \frac{(1-\beta)\phi_y}{\kappa}$ *and* $\phi_\pi^2 = \phi_\pi^1 + \frac{\beta}{4\sigma\kappa}\left(1-\frac{1}{\beta}-\sigma\phi_y-\frac{\sigma\kappa}{\beta}\right)^2$*.*

Proof. See Appendix A. ■

In other words, when the response of the monetary authority to inflation $\phi_\pi < \phi_\pi^1$ is weak, we have one stable and one unstable root; when the response to inflation becomes stronger, both roots become unstable $\phi_\pi^1 \leq \phi_\pi < \phi_\pi^2$; when the response to inflation reaches the threshold level ϕ_π^2, the roots become repeated and unstable and finally; when $\phi_\pi > \phi_\pi^2$, the roots are complex and unstable.

Woodford (2001) calculated the boundary ϕ_π^1 of Theorem 1. Cochrane (2011) derived stability conditions under several Taylor rules with leads and lags, including $i_t = \phi_\pi\pi_t$, $i_t = \phi_{E\pi}E_t[\pi_{t+1}]$, and $i_t = \phi_\pi\pi_t + \phi_y x_t$ (see his Appendix B, Section E). Our Theorem 1 and its corollary provide sharper results by establishing boundaries that separate different types of roots, namely, distinct real roots, repeated real roots and complex roots. These results are a useful step in constructing closed-form solutions since different types of roots lead to different types of solutions.

2.3 Closed-form solutions

We now show closed-form solutions to the model (1)–(3) under four possible cases of characteristic roots established in Theorem 1.

Theorem 2. *A solution for inflation* π_t *in the new Keynesian model (4) for cases i)-iv) in Theorem 1 is* *given by the sum of a general and particular solutions,*

Case	Restriction on roots	Solution for π_t General solution	Particular solution
i)	$\lvert m_1\rvert \geq 1,\ \lvert m_2\rvert < 1$	$C_1 m_1^t + C_2 m_2^t$	$\frac{1}{m_1-m_2} E_t \left[\sum_{s=t}^{\infty} m_1^{t-1-s} z_s + \sum_{s=-\infty}^{t-1} m_2^{t-1-s} z_s\right]$
ii)	$\lvert m_1\rvert \geq 1,\ \lvert m_2\rvert \geq 1$	$C_1 m_1^t + C_2 m_2^t$	$\frac{1}{m_1-m_2} E_t \left[\sum_{s=t}^{\infty} m_1^{t-1-s} z_s - \sum_{s=t}^{\infty} m_2^{t-1-s} z_s\right]$
iii)	$m_1 = m_2 = m$ $\lvert m\rvert > 1$	$(C_1 + C_2 t)\, m^t$	$\frac{1}{m} E_t \left[(t-1) \sum_{s=t}^{\infty} m^{t-1-s} z_s - \sum_{s=t}^{\infty} s m^{t-1-s} z_s\right]$
iv)	$m_{1,2} = \mu \pm \eta\iota$ $r \equiv \sqrt{\mu^2 + \eta^2} > 1$ $\theta \equiv \arctan\left(\frac{\eta}{\mu}\right)$	$C_1 r^t \cos(\theta t) + C_2 r^t \sin(\theta t)$	$\frac{1}{\eta} E_t \left[\sum_{s=t}^{\infty} r^{t-1-s} \sin(\theta(t-1-s))\, z_s\right]$

Furthermore, in case i), there are multiple forward stable solution characterized by $C_1 = 0$ and arbitrary C_2; and in cases ii)–iv), there is a unique forward stable solution characterized by $C_1 = 0$ and $C_2 = 0$.

Proof. The solution to (4) is given by the sum of a general solution to homogeneous equation $\pi_{t+2} + b\pi_{t+1} + c\pi_t = 0$ and a particular solution satisfying the non-homogeneous equation (4). Homogeneous second-order difference equations with constant coefficients are well studied in the field of differential equations; the solutions to such equations contain integration constants C_1 and C_2. In contrast, there is no general approach that can deliver particular solutions to the studied non-homogenous equations. One possible approach is a version of the "trial and error" method that parameterizes a particular solution by a linear combination of two terms in the solution to the homogeneous equations and identifies the coefficients in the combination to satisfy the given non-homogeneous equation – this is the approach we used in the paper.[6] The fact that the constructed particular solutions satisfy the non-homogeneous equation can be verified directly, by substituting them into (4).

Concerning forward stability, note that the particular solutions in Theorem 2 are forward stable (non-explosive) by construction. To be specific, if a root m_i is unstable, i.e., $m_i > 1$, we use a particular solution that is forward looking $\sum_{s=t}^{\infty} m_i^{t-1-s} z_s$, while if it is stable, i.e., $m_i < 1$, we use the one that is backward looking $-\sum_{s=-\infty}^{t-1} m_i^{t-1-s} z_s$. To make the entire solution stable, the solution to the homogeneous equation must also be forward stable. In cases *ii)-iv)*, this requires us to set the integration constants at $C_1 = 0$ and $C_2 = 0$. However, in case i), stability is consistent with any integration constant C_2 on the stable root m_2. Therefore, the stable solution is unique in cases ii)-iv), and there is a multiplicity of forward stable solutions in case i). ■

Closed-form solutions to the cases *ii)-iv)* are new to the literature (to the best of our knowledge). Case *i)* generalizes a closed-form solution that was previously derived in Cochrane (2017a, 2017b) for a version of the model in which all the coefficients in Taylor rule are set at zero because the monetary authority follows a pre-specified interest-rate path. Theorem 2 shows that such solution also applies to the economies in which Taylor rules are insufficiently active. Finally, Maliar (2018) constructs parallel solutions i)-iv) for the continuous-time version of the model (1)–(3).

In the remaining paper, we use the constructed closed-form solutions to study the effectiveness of FG. We focus on conventional equilibria that do not explode in the future – forward stable equilibria. We model FG as an exogenous future policy shock, unrelated to the systematic reaction of the monetary authority to inflation and the output gap and anticipated by agents (but not necessarily announced). In periods other than those affected by FG, the monetary policy is given by a flexible Taylor rule which is left free to react to endogenous variables. For the sake of presentational convenience, we focus on a single FG

[6] To construct the particular solutions in the cases i) and ii), we can also use the approach of Cochrane (2017a) of decomposing the second-order difference equation into two first-order difference equations, however, this approach does not directly apply to cases iii) and iv).

shock, although our closed form solutions apply to arbitrary sequence of shocks: Theorem 2 shows that the impacts of contemporaneous and future shocks are aggregated with the weights that depend on the values of the characteristic roots.

Our Theorems 1 and 2 delineate the parameter space into four regions characterized by four different types of the characteristic roots (or eigenvalues). The coefficients of the Taylor rule – the responsiveness of the interest rate to output, inflation, and expected inflation—are the key determinants of the distinguished regions. Our results provide a simple way to see weather or not a given parameterization leads to the FG puzzle, namely, it is sufficient to check to which region it belongs. Specifically, the region (i) has a stable root and such root produces a backward explosion and FG puzzle; and the regions (ii), (iii) and (iv) have two unstable roots (either real or complex) which ensures backward stability so the FG puzzle is not observed. An exception is the borderline case of a unit root between the regions (i) and (ii) in which a weaker form of the FG puzzle is observed. We next analyze the constructed parameter regions in more details.

3 Choosing the interest-rate path: indeterminacy and FG puzzle

In this section, we analyze the region i) in which is the model that has one stable (less-than-one) and one unstable (large-than-one) root, and as a result, has multiple forward stable solutions. A limiting case of this region is the Taylor rule (3) in which all the coefficients are equal to zero, i.e., $\phi_\pi = 0$, $\phi_y = 0$, $\phi_{E\pi} = 0$. In terms of our model, this case corresponds to the analysis of Werning (2015) and Cochrane (2017a) who consider a model with two equations (the IS and Phillips curves) and two unknowns $\{x_t, \pi_t\}$, and who assume that monetary authority does not follow any rule but chooses the sequence of interest rates $\{i_0, i_1, ...\}$ directly. These papers argue that the policy of the pre-specified interest-rate path approximates well the monetary policy in the post financial crisis period, characterized by nominal interest rates being at the ELB, first, due to the binding ELB and then, due to FG policy.

An important question is how the agents perceive the announced interest rate path. Cochrane (2017a) argues that a given interest rate path defines the Taylor rule implicitly in the sense that given $\{x_t, \pi_t, i_t\}$, the agents can infer the implied coefficients ϕ_π, $\phi_{E\pi}$ and ϕ_y in (3). In our case, rational agents must infer that these coefficients are all zeros, implying that the Taylor rule is completely passive. That is, no matter what shock occurs in the future, the monetary authority commits to do nothing to offset the shock and and maintains the interest-rate path. (If the monetary authority reacts to a shock, the inference of agents about zero feedback coefficients would be incorrect).

By Theorem 2, case i), a closed-form solution for $|m_1| \geq 1$ and $|m_2| < 1$ takes the form

$$\pi_t = C_2 m_2^t + \frac{1}{m_1 - m_2} E_t \left[\sum_{s=t}^{\infty} m_1^{t-1-s} z_s + \sum_{s=0}^{t-1} m_2^{t-1-s} z_s \right], \tag{6}$$

where C_2 is an arbitrary constant. In particular, for a single anticipated FG shock at T, we have $\varepsilon_t = 0$ for $t \neq T$, and $\varepsilon_T = \varepsilon$, so that $z_t = 0$ for $t \neq T$, and $z_T = \frac{\kappa\sigma\varepsilon}{\beta}$. Substituting the latter result into the solution (6), we get the following impulse-response to the FG shock:

$$t \leq T,\ \pi_t = C_2 m_2^t + \frac{\kappa\sigma\varepsilon}{\beta (m_1 - m_2)} m_1^{t-1-T}; \tag{7}$$

$$t > T,\ \pi_t = C_2 m_2^t + \frac{\kappa\sigma\varepsilon}{\beta (m_1 - m_2)} m_2^{t-1-T}. \tag{8}$$

That is, the economy is driven by a forward-looking component $\sum_{s=t}^{\infty} m_1^{t-1-s} z_s$ before the shock occurs and it is driven by a backward-looking component $\sum_{s=-\infty}^{t-1} m_2^{t-1-s} z_s$ afterwords. Since stability is consistent with any C_2, we can choose it in an arbitrary manner.

Equations (7), (8) illustrate the impact of the forward guidance on the economy with one stable and one unstable root on which much of the FG literature related to the ELB scenario focuses. Such solutions converge forward and diverge backward. The further are the policy announcements in the future, the larger are their effects on the today's economy. This result is the classic *forward guidance puzzle* – an observation that in a stylized new Keynesian model with ELB, output and inflation react excessively and unrealistically to central bank's announcements about future interest rates changes. We can also see that the quantitative expression of the FG puzzle is a smooth function of m_2.

As an illustration, we plot the impulse responses (7), (8) for $T = 30$ in Figure 1. To produce this and all subsequent solutions, we parameterize the model by $\kappa = 0.11$, $\sigma = 1$ and $\beta = 0.99$, which produces eigenvalues of $m_1 = 1.4052$ and $m_2 = 0.7188$.[7] We display two stable equilibria: in one of them, we chose $C_2 = 0$, and in the other, we chose C_2 such that initial inflation is zero, i.e., $\pi_0 = 0$.

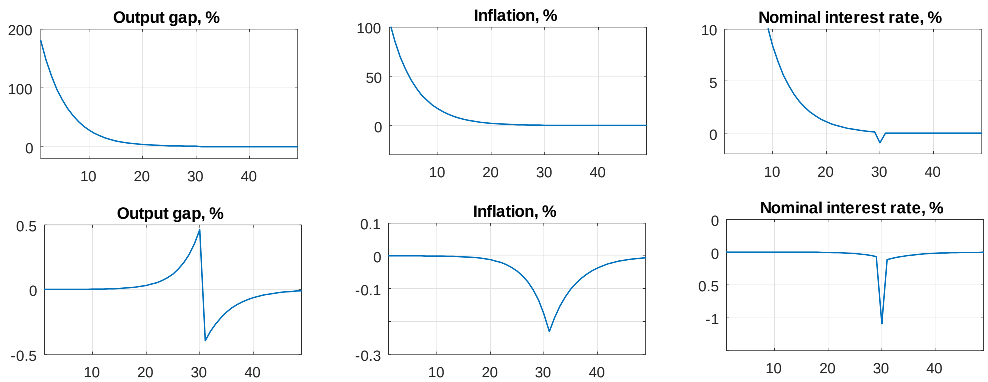

Figure 1. Multiple equilibria under $\phi_{E\pi} = 0$, $\phi_\pi = 0$, $\phi_y = 0$: equlibria with $C_2 = 0$ (top panels) and with $\pi_0 = 0$ (bottom panels)

Figure 1 shows that the effect of FG on the economy depends critically on the equilibrium selection (i.e., on the integration constant C_2), ranging from extremely large (equilibrium with $C_2 = 0$) to practically non existent (equilibrium with $\pi_0 = 0$). Similarly, the effect on inflation is dramatic in the former case, and it is very mild in the latter case. Thus, the upper panel of the figure is a classical FG puzzle response. In the lower panel, the monetary authority manages to coordinate on equilibrium which involves no backward explosion.

In the context of a model with a fixed interest rate path, the FG literature argues that some equilibrium choices make more sense than others. To see the point, suppose there is a transitory IS disturbance that lasts from period 0 to T (e.g., the real interest rate happens to be negative in this interval of time). Werning (2012) selected an equilibrium with $\pi_{T+1} = 0$, arguing that people will expect it because of the forward-looking optimality. This choice of inflation implies a specific value of C_2 and $\pi_0 \neq 0$. The resulting solution generically explodes backwards and therefore, implies the FG puzzle. Similar equilibrium selection is implemented in Carlstrom et al. (2015) by solving the model backward starting from a given terminal conditions. Their models also generate a backward explosion and the FG puzzle.[8] In contrast, Cochrane (2017a) proposed to resolve the FG puzzle by setting C_2 in the way that ensures $\pi_0 = 0$. To justify this equilibrium, he noted that $\pi_0 < 0$ represents an unexpected deflation that induces an increase in the value of government debt which requires fiscal tightening to pay off. Absent such fiscal policy, we have $\pi_0 = 0$, which, as argued above, resolves the puzzle.

[7] The value of $\kappa = 0.11$ corresponds to a fraction of non-reoptimizing firms $\theta = 0.83$ and a utility-function parameter $\vartheta = 2.09$; see our footnote 8 for the formula of κ.

[8] In the medium-scale DSGE models of Christiano et al. (2005) and Smets and Wouters (2007), Carlstrom et al. (2015) use the same equilibrium selection approach and finds diverse effects of FG – from none to very large – depending on the number of FG periods and on the presence of inflation indexation. These effects seem to follow a very complex, sometimes "nonsensical" structure.

The novelty of our formal analysis is that it puts the experiments in the previous literature as particular examples of the region i). It shows that the large differences between multiple equilibria discovered by Werning (2012) and Cochrane (2017a) are not limited to the policy of the pre-specified interest rate path but are a generic property of the new Keynesian model in which the monetary policy is not sufficiently responsive to insure the equilibrium uniqueness. Our Theorem 2 implies that any Taylor rule within the region i) is characterized by one stable and one unstable root and has qualitatively similar predictions to the model with a fixed interest rate path.

4 Accommodating inflation: equilibrium uniqueness and FG puzzle

When the degree of responsiveness of the Taylor rule increases, we arrive to region ii), specifically, we attain the turning point when the smallest root m_2 reaches a unit size and the equilibrium becomes unique. It is shown in McKay et al. (2016) that this boundary case also leads to a FG puzzle, although the effect of the future shocks on the economy is less strong. The Taylor rule they consider is $i_t = i_t^n + E_t[\pi_{t+1}] + \varepsilon_t$ (this is a specific case of our general Taylor rule (3) under $\phi_{E\pi} \searrow 1$, $\phi_\pi = 0$ and $\phi_y = 0$). The characteristic roots are $m_1 = \frac{1}{\beta}$ and $m_2 \searrow 1$. As is pointed out by McKay et al. (2016), the considered rule is inactive, specifically, they write "To build intuition, we have assumed that there is no endogenous feedback from changes in output and inflation back onto real interest rates". Therefore, this Taylor rule corresponds to the case of a fixed real interest-rate path. In a sense, this case similar to the previously studied case of fixed nominal interest rate in Section 3.

To characterize this case with our closed-form solution, let us assume that all shocks are zero, except of the shock at T which is equal to $z_T = \frac{\kappa\sigma\varepsilon}{\beta}$. By Theorem 2, for $t \leq T$, the solution is given by

$$\pi_t = \frac{1}{1/\beta - 1}\left[\sum_{s=t}^{\infty}\left(\frac{1}{\beta}\right)^{t-s-1} z_s - \sum_{s=t}^{\infty} z_s\right] = \frac{\kappa\sigma\varepsilon}{1-\beta}\left[\beta^{T-t+1} - 1\right], \tag{9}$$

and $\pi_t = 0$ for $t > T$. From the Phillips curve (1), we have $x_t = \frac{1}{\kappa}(\pi_t - \beta\pi_{t+1}) = -\sigma\varepsilon$ for $t \leq T$ and $x_t = 0$ for $t > T$. This means that, a shock that will happen in any remote period T has the same effect on current output x_t as the one that happens at present. The impact of future shocks on inflation is even more dramatic: the further away the shock is in the future, the larger is its effect on today's inflation, as (9) shows. Note however that the inflation dynamics is not explosive backward but converges to a limit $\lim_{t\to-\infty} \pi_t = -\frac{\beta}{1-\beta} z_T = -\frac{\kappa\sigma\varepsilon}{1-\beta}$.

Figure 2 illustrates the FG puzzle graphically. The figure plots the output gap, inflation and nominal interest rate in response to a one-percent negative shock to the nominal interest rate that happens in the 30th quarter, $T = 30$.

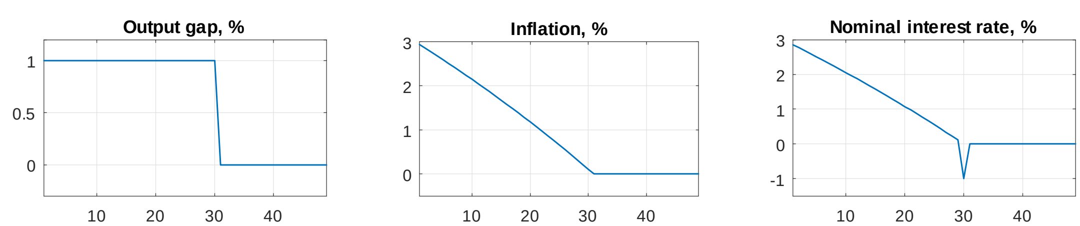

Figure 2. Taylor rule with expected inflation, $\phi_{E\pi} \searrow 1$.

The output gap goes up immediately by one percent in response to a distant future shock. The inflation goes up by about three percent, and then gradually decreases and reaches the steady state level in period

30. A promised future fiscal stimuli and capital destruction will have the same magical effect. This is because a positive monetary policy shock acts similarly to a disturbance to government spending.

Why are the future shocks so powerful? As was pointed out by Del Negro et al. (2012, 2015) and McKay et al. (2016), this happens because the effect of future shocks on output is not discounted. To see the point, let us apply forward recursion to the IS curve (1) by imposing a forward stability condition $\lim_{s\to\infty} x_{t+s} = 0$ (a steady-state value) and let us assume again the monetary policy rule $i_t = i_t^n + E_t[\pi_{t+1}] + \varepsilon_t$. We get

$$x_t = -\sigma \sum_{s=t}^{\infty} (i_s - i_s^n - E_s[\pi_{s+1}]) = -\sigma \sum_{s=t}^{\infty} \varepsilon_s. \tag{10}$$

(In the case of a single T-period shock, (10) leads to the same expression as our closed-form solution $x_t = -\sigma\varepsilon$). In formula (10), there is no discounting and all shocks affects the output in the same manner. The discounting is absent because the monetary authority commits to fully accommodate the inflationary expectations. It does so by increasing the nominal rate to maintain the target real rate. In the absence of stabilization, such increase propagates backward. Notice, however, that the impact of future shocks on the economy is less dramatic than the one in the economy with pre-specified interest rate path in Figure 1. This is because the Taylor rule with a unit coefficient in Figure 2 implies certain degree of stabilization while the pre-specified interest-rate path is completely irresponsive and implies no stabilization at all.

An interesting question is: How different would the results be if the monetary rule (3) would depend on actual rather than expected inflation, i.e., $\phi_{E\pi} = 0$, $\phi_\pi = 1$ and $\phi_y = 0$ leading to $i_t = i_t^n + \pi_t + \varepsilon_t$ which is another case with a unit root, namely, $m_1 = \frac{1+\sigma\kappa}{\beta} > 1$ and $m_2 \searrow 1$? A closed form solution for inflation is given by (9)

$$\pi_t = \frac{\kappa\sigma\varepsilon}{1-\beta+\sigma\kappa}\left[\left(\frac{\beta}{1+\sigma\kappa}\right)^{T-t+1} - 1\right], \tag{11}$$

where the last expression corresponds to the case of a single shock $z_T = \frac{\kappa\sigma\varepsilon}{\beta}$. From the Phillips curve, the corresponding solution for output is $x_t = \frac{-\sigma\varepsilon}{1-\beta+\sigma\kappa}\left(1-\beta+\frac{\beta\sigma k}{1+\sigma\kappa}\left(\frac{\beta}{1+\sigma\kappa}\right)^{T-t}\right)$. If the shock is distant, i.e., $T \gg t$, so that $\left(\frac{\beta}{1+\sigma k}\right)^{T-t} \approx 0$, the future shock increases output initially to $x_0 \approx \frac{-\sigma\varepsilon}{1-\beta+\sigma\kappa}(1-\beta) \approx -0.08\sigma\varepsilon$ and continues to raise it till it reaches $x_T = \frac{-\sigma\varepsilon}{1-\beta+\sigma\kappa}\left(1-\beta+\frac{\beta\sigma\kappa}{1+\sigma\kappa}\right) \approx 0.9\sigma\varepsilon$ at T (assuming $\kappa = 0.11$, $\sigma = 1$ and $\beta = 0.99$). This case is illustrated in Figure 3.

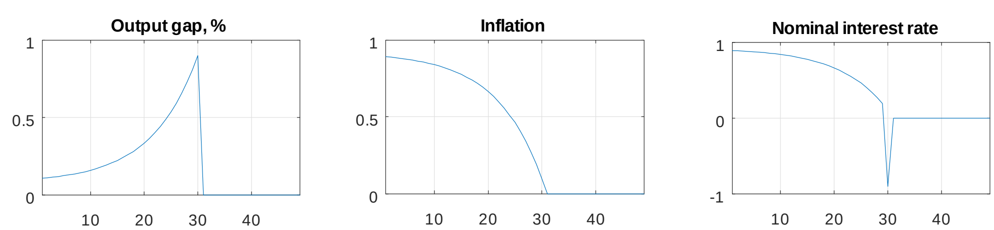

Figure 3. Taylor rule with actual inflation, $\phi_\pi = 1$.

Thus, we observe a weaker version of the FG puzzle than the one analyzed in McKay et al. (2016). The initial response of inflation to anticipated interest rate shock is large but the initial response of output is modest (it is just 8 percent of what we had in the example of McKay et al. (2016)). As the economy advances and approaches the period 30, the impact of the shock on output gradually increases to reach

90 percent of the one in McKay et al. (2016). The effect of FG is even less strong on Figure 3 than on previous Figures 1 and 2 because the Taylor rule with actual inflation implies larger unstable root m_1 and hence, a higher degree of stabilization.

5 Active Taylor rules: equilibrium uniqueness and backward stability

When we increase the responsiveness of the Taylor rule even further, both characteristic roots become unstable (either real or complex). This is true everywhere in the regions ii), iii) and iv) with an exception of the unit root borderline case of region ii) discussed in Section 4. We now have both a unique equilibrium and backward stability, i.e., the effect of future policy announcements implodes backward to zero, and the FG puzzle is not observed. The condition that $\phi_\pi > 1$ is referred to as the *Taylor Principle.* Recent Monetary Policy Reports of the Fed focus on this case: "Policy rules can incorporate key principles of good monetary policy. One key principle is ... . A third key principle is that, to stabilize inflation, the policy rate should be adjusted by more than one-for-one in response to persistent increases or decreases in inflation." (see the reports from July 7, 2017, February 23, 2018, July 17, 2018). In fact, we will show that backward stability is obtained not only when the inflation or expected inflation coefficients increase but also when the Taylor rule includes the output gap stabilization.

Taylor rule with the output gap and anticipated inflation. Let us consider the rule (3) with anticipated inflation $\phi_{E\pi} > 0$ and the output gap $\phi_y > 0$. Substituting the Taylor rule (3) into the IS curve (1), we obtain $x_t = \frac{1}{1+\sigma\phi_y}\left[E_t x_{t+1} - \sigma\left(\phi_{E_\pi} - 1\right) E_t\pi_{t+1} - \sigma\varepsilon_t\right]$. By making recursive substitution and by imposing $\lim\limits_{s\to\infty} x_{t+s} = 0$, we get

$$x_t = -\left(\phi_{E_\pi} - 1\right)\widetilde{\beta}\sigma\sum_{s=t}^{\infty}\widetilde{\beta}^{s-t}E_s\pi_{s+1} - \widetilde{\beta}\sigma\sum_{s=t}^{\infty}\widetilde{\beta}^{s-t}\varepsilon_s, \tag{12}$$

where $\widetilde{\beta} \equiv \frac{1}{1+\sigma\phi_y}$ is an effective discount factor. Since $\phi_y > 0$, the effect of future shocks on today's output is discounted at the rate $\widetilde{\beta} < 1$, unlike under the FG puzzle (10). In particular, in the benchmark case $\phi_{E\pi} \searrow 1$, we obtain $x_t = -\widetilde{\beta}\sigma\sum_{s=t}^{\infty}\widetilde{\beta}^{s-t}\varepsilon_s$, which is identical to the FG-puzzle formula (10) (up to the multiplicative term) except that now we have discounting. Thus, we proved that it is sufficient to introduce the output stabilization to eliminate the backward explosion and the FG puzzle.

Why the introduction of more active Taylor rules leads to discounting of future shocks? This is because the agents know that if the shocks occur in the future, the monetary authority will intervene to stabilize the economy, reverting back to the steady state. Thus, when evaluating the effect of the future shocks on today's economy, the agents discount the shocks by the expected interventions of monetary authority. Recall that it was not the case with a fixed interest rate path and the policy of accommodating inflation.

Taylor rule with the output gap and actual inflation. Alternatively, we can consider the Taylor rule with actual inflation $\phi_\pi > 0$ and the output gap $\phi_y > 0$. Substituting the t-period Taylor rule (3) into (1) and imposing (2), we get $x_t = \frac{\beta}{\beta+\sigma\kappa+\sigma\beta\phi_y}x_{t+1} - \frac{\beta\sigma}{\beta+\sigma\kappa+\sigma\beta\phi_y}\left(\phi_\pi\pi_t - \frac{\pi_t}{\beta} + \varepsilon_t\right)$. Again, by using a forward recursive substitution of the future output gaps and by imposing $\lim\limits_{s\to\infty} x_{t+s} = 0$, we obtain

$$x_t = -\left(\phi_\pi - \frac{1}{\beta}\right)\overline{\beta}\sigma\sum_{s=t}^{\infty}\overline{\beta}^{s-t}\pi_s - \overline{\beta}\sigma\sum_{s=t}^{\infty}\overline{\beta}^{s-t}\varepsilon_s, \tag{13}$$

where $\overline{\beta} \equiv \frac{\beta}{\beta+\sigma\kappa+\sigma\beta\phi_y} < 1$ is an effective discount factor. Like in the previous case, we have $\overline{\beta} < 1$, so that the effect of the shock ε_s on today's output is discounted at the rate $\overline{\beta}$. Furthermore, in a special case $\phi_\pi = \frac{1}{\beta}$, we obtain $x_t = -\overline{\beta}\sigma \sum_{s=t}^{\infty} \overline{\beta}^{s-t} \varepsilon_s$, which is again identical to the FG-puzzle formula (10) except for the presence of discounting (and the multiplicative term $\overline{\beta}$). Interestingly, discounting does not disappear even if $\phi_y = 0$ since we have $\overline{\beta} = \frac{\beta}{\beta+\sigma\kappa} < 1$ (as long as $\kappa > 0$). However, a larger output gap in the Taylor rule (3) makes discounting stronger.

A comparison of the Taylor rules with anticipated and actual inflation. In Figure 4, we compare the results under two Taylor rules (3) that both contain an identical output gap coefficient $\phi_y = 0.5$ but one rule contains just expected inflation $\phi_{E\pi} = 2$ (see a red line), while the other contains just actual inflation $\phi_\pi = 2$ (see a blue line). These are the conventional values used in the related literature, see, e.g., Taylor (1993), and Coibion et al. (2012). We used the constructed closed form solutions, to generate the series for the output gap, inflation and the interest rate for these two cases.

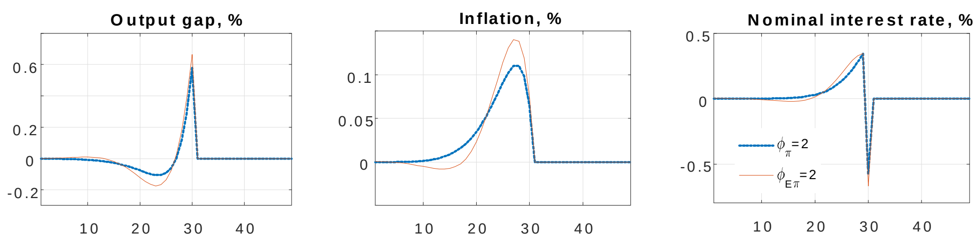

Figure 4. Taylor rules with the output gap $\phi_y = 0.5$: a comparison of anticipated inflation , $\phi_{E\pi} = 2$, and actual inflation, $\phi_\pi = 2$.

In the figure, the two Taylor rule's parameterizations lead to similar qualitative behavior of the model's variables. This similarity can be understood by realizing that our decompositions (12) and (13) imply virtually identical formulas for x_t except of the effective discount factor. The FG puzzle is not observed: the effect of a distant shock on today's variables is negligible, i.e., inflation and output are backward stable. Thus, the plausible Taylor rules insure both backward stability and equilibrium uniqueness.

Finally, it turns out that for our calibration, the considered parameterization of the Taylor rule is close to the borderline that separates the case of unstable real roots of regions ii) and that of the complex roots of region iv). In particular, $\phi_{E\pi} = 2$ or $\phi_\pi = 2$ shown in Figure 3 lead to complex roots and belong to region iv) and, while $\phi_{E\pi} = 1.5$ or $\phi_\pi = 1.5$ lead to real roots of region ii). Qualitatively, dynamics in both cases look very similar; we show the case of the real roots in Figure D1 of Appendix D. The latter parameterization is distinguished in Taylor (1993) as the most plausible one.

The speed of convergence and the FG puzzle As Theorem 2 implies, all solutions constructed under two unstable roots in the regions ii)-iv) are backward stable. However, this result is asymptotic. In applications, it is also important to know how rapidly a solution converges backward. In particular, if convergence is very slow, the effect of distant shocks can be still unrealistically large, which is observationally equivalent to the FG puzzle.

In particular, the original definition of the FG puzzle was coined in quantitative rather than asymptotic sense in the work of Del Negro et al. (2012, 2015). They conduct a policy experiment that consists in Fed announcement to maintain the rates fixed at 0.25 for a given number of periods before following the historical Taylor rule. Del Negro et al. (2015) highlights an excessive power of FG in the context of FRB NY DSGE model. As is shown in their Figure 4, the model predicts that as a result of a policy announcement, real GDP growth increases to 3.5% in 2012 and 4.9% in 2013, while inflation jumps to

1.8% and 1.9%, respectively. These excessively large responses of output and inflation to a small shock is precisely what Del Negro et al. (2015) call the FG puzzle.[9]

To illustrate the importance of the speed of backward convergence, in Figure 5, we show the solution under two Taylor rules: one rule reacts to actual inflation $\phi_\pi = 3$ and the other rule reacts to the expected inflation $\phi_{E\pi} = 3$. (In Figures D2 and D3 of Appendix D, we provide further illustration of these solutions).

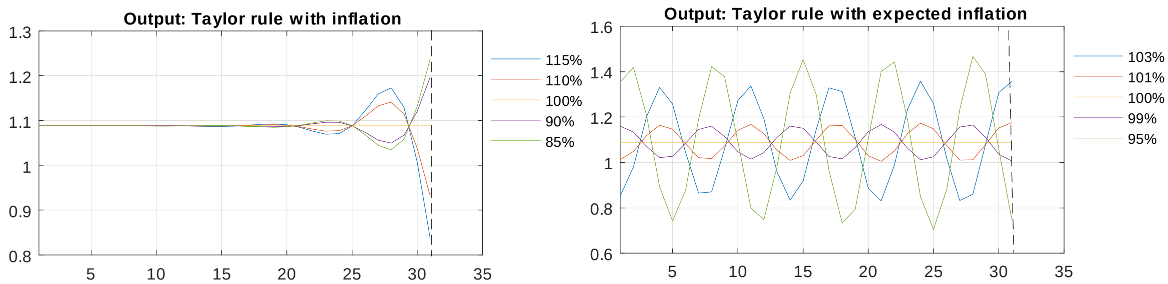

Figure 5. The effect of variations in the terminal condition under the Taylor rules with $\phi_\pi = 3$ and $\phi_{E\pi} = 3$.

In both models, the roots are in the complex region iv), so by Theorem 2, they are backward stable. While the model with actual inflation in the figure shows a rapid backward convergence pattern, the model with expected inflation appears to have a cycling non-convergent pattern. In the latter model, the root is close to unity: as a result, the fluctuations decay so slowly that they appear to be nonvanishing. For the right panel in Figure 5, we can conclude that there is a FG puzzle in the sense of the definition of Del Negro et al. (2015).

Generality of our results We assess the generality of our findings. In the main text, we focus on the deterministic case in which the sequence of shocks $\{z_t, z_{t+1}, ...\}$ is known at t. Since the agents have perfect foresight, we omit the expectation operator in Theorems 1 and 2. In Appendix B, we show that our deterministic solutions can be generalized to the case of uncertainty, including temporary and permanent shocks, anticipated and unanticipated shocks, as well as various mixtures of deterministic trends and stochastic shocks. Another method that allows to deal with uncertainty is the method of undetermined coefficients proposed by Taylor (1986) but that method does not specify how to construct a solution to the deterministic model like those we obtain in Theorem 2; see Appendix B for a comparison of the two methods.

Also, in the paper, we restrict attention to monetary-policy anticipated shocks, however, the conclusions of our analysis carry over to other types of anticipated shocks, e.g., those to future technological improvements. Our closed-form solutions imply that the impact of a technology shock would generate the same sort of FG patterns as those discussed in Sections 3-5. That is, a technology shock, anticipated to happen in the future, will immediately change output and inflation today. Similar impacts will result from changes in direct marginal costs (due to capital destruction, technical regress, government spending) or in the natural rate of interest; the latter would allow us to study secular stagnation. Our results also imply that there are many other combinations of the parameters $\phi_{E\pi}$, ϕ_π and ϕ_y in the Taylor rule that lead to a unit root and produce some versions of the FG puzzle. Our Theorems 1 and 2 make it possible to establish all such parametererizations and to show the corresponding closed-form solution for each of them.

Finally, we use numerical simulation to explore the properties of equilibria under more general assumptions when closed-form solutions are infeasible. We show that the results we establish analytically are

[9] The FG puzzle is not a generic feature of medium-scale NK models. Campbell et al. (2016) find that their medium-scale new Keynesian model estimated with US data differs from that in Del Negro et al. (2015) in several dimensions, and it produces realistic responses for empirically plausible interest-rate pegs.

robust to the modifications. In particular, they hold under more general Taylor rules that include lagged interest rate, in addition to inflation and output gap. Also, they hold in a non-linear version of the studied new Keynesian model, as well as in the version of such model augmented to include capital. In addition, our findings are robust to the solution methods used, specifically, we find that both Fair and Taylor's (1983) extended path method and Maliar et al.'s (2020) extended function path method lead to similar results. These additional results are elaborated in Appendices D and E.

6 The power of the open-mouth policy

There are two main definitions of FG in the literature. For example, Bean (2013), then deputy governor of the Bank of England, states that FG "is intended primarily to clarify our reaction function," which is to give a description of how the policy interest rate will react to economic variables, or a monetary policy rule. According to that definition, FG is simply an announcement that monetary policy follows a policy rule now and in the future. This type of FG reduces uncertainty, helping the agents to plan their economic actions more efficient. Alternatively, Reifschneider and Williams (2000), and Woodford (2013) define FG as an announced deviation from the policy rule; they consider such a deviation in the context of the ELB with the goal to produce a positive stimulating effect on the economy. We are interested in the effectiveness of FG in the latter sense, i.e., we aim to analyze whether or not FG can be used as a policy instrument for producing a positive impact on the economy. Our analysis had shown that the answer to the question depends on how active the monetary policy is in those periods that are not affected by the FG announcement. The cases of active and passive monetary policies are discussed below in more details.

FG with active monetary policy. While FG became popular at the time of binding ELB during and after the Great Recession, it was used before that time by the Fed and other central banks. In particular, there is evidence that the Fed was using FG in 2003-2004 before the ELB actually happened; see Carlson et al. (2008) and Plosser (2013) as well as in the non-ELB periods; see Gürkaynak et al. (2005) and Campbell et al. (2012). Furthermore, there is also an interest among central bankers in using FG in the future. Specifically, out of 55 heads of central banks, surveyed in Blinder et al. (2016), none said that FG should be discontinued after the crisis; 59% and 12.8% think that it is a potential instrument in the same and modified form, respectively.

There is also a voice calling for FG in normal times among policymakers. Bernanke (2017) argues that FG can be useful before the next recession hits, by noting that "... when ZLB looms, rate cuts should be aggressive ... Forward guidance, of the Odyssean variety, would come next ... . Relative to earlier experience, I would expect a much earlier adoption of state-contingent, quantitative commitments to hold rates low." The former Fed's chair, Yellen (2018) has a similar opinion by arguing that " the FOMC should seriously consider pursuing a lower-for-longer or makeup strategy for setting short rates when the zero lower bound binds and should articulate its intention to do so before the next zero lower bound episode". Mester (2014) views FG as a device that in normal time "conveys to the public how policy is likely to respond to changes in economic conditions"; Coeuré (2018) also supports the usefulness of FG "beyond the timing to lift-off", etc.

Our analysis helps understand whether or not FG is a potentially useful monetary policy tool when the ELB does not bind. It shows that if the central bank following an active Taylor rule can undo the effect of the FG commitments, including the backward explosive effects characteristic for the FG puzzle. But then why would the central bank announce a FG shock in the first place just to undo it with its future actions? Thus, our analysis does not provide a basis for using FG for stimulating the economy in normal times.

FG with passive monetary policy. Let us consider the power of FG combined with passive systematic monetary rules. Passive policies do not provide stabilization that is sufficient to "undo" the effect of future shocks. In particular, the FG puzzle shows policy announcements that can have huge positive effects on the economy. So, why cannot the monetary authority take advantage of the open-mouth policy by using passive monetary rules that lead to backward explosion, in particular, the policy of the pre-specified interest rate path?

One possible reason for doubting the effectiveness of FG is pointed out by Cochrane (2017a) who argues that in the presence of multiple equilibria, the agents can coordinate on a specific equilibrium in which the FG puzzle is absent. However, there are also passive policy rule with a unit root that lead to unique equilibrium and still imply large stimulating FG effects, for example, the rule studied in McKay et al. (2016). Maybe policymakers can use these rules for stimulating the economy.

In fact, nothing in our analysis prevents the monetary authority from using these rules. However, there is a question whether the model parameterized by such rules represents a meaningful representation of reality. One of the fundamental principals of monetary theory is that the central bank aims to maximize social welfare. A coherent way to model the central bank's problem is to construct a Ramsey solution. The socially optimal inflation implied by the Ramsey problem is zero, so the Ramsey policy involves an active inflation stabilization. Woodford (2001) shows that the Taylor rules provide a good approximation to the Ramsey policy rule.[10] But this is true not for all Taylor rules but only for those that satisfy the Taylor principle and insure the active inflation stabilization.

However, the Taylor rule can differ dramatically from the Ramsey rule under some parameterizations. For example, the rule $i_t = i_t^n + E_t[\pi_{t+1}] + \varepsilon_t$ studied in McKay et al. (2016) implies that the monetary authority will set the nominal interest rate to fully accommodate the inflation expectations. Let's say, if the people expect 300 percent inflation, the monetary authority will do nothing but set 302 percent nominal interest rate to guarantee the real interest rate of 2 percent. A policy of the pre-specified interest rate path is even more extreme: the monetary authority commits to stick to the announced path no matter how devastating the shock is. Such policies contrast sharply with the Ramsey optimal prescription of inflation stabilization and may lack credibility. The agent knows that if necessary, the monetary authority will revert the FG commitments or it can use other tools to "undo" the effects of such commitments if that become welfare reducing, so passive policies have limited credibility.

If the central bank does not maximize the social welfare but does something else, the model in the paper predicts a variety of empirically implausible outcomes, including the FG puzzle. The fact that suboptimal policy rule lead to counterintuitive and counterfactual implications is not entirely surprising: one would expect equally puzzling implications from the models in which consumers do not maximize utilities or firms do not maximize profits. It is not clear if the predictions of such models are meaningful. A reasonable approach for the monetary authority would be to aim at attaining an optimal outcome by implementing a Taylor-style rule that approximates Ramsey policy. Such an approach does not lead to the FG puzzle, as our theorems show. If active Taylor-style rules are not available, e.g., because of binding ELB, the monetary authority is likely to find other systematic policy rules that approximate the Ramsey solution, possibly by involving the fiscal policy and other instruments. Such systematic policy rules are unlikely to leave much room for the FG announcements.

7 Conclusion

In the paper, we accomplished four main goals: First, we explain what produces the FG puzzle. We show that even though the FG puzzle is related to the ELB, it is not the ELB or new Keynesian model per se that produce the explosive backward behavior, but the imposed passivity of the monetary policy rule, not only in the periods where the ELB is binding, but also in the periods where the shadow rate would be normally above the ELB.

Second, we show how to effectively characterize the FG puzzle. We demonstrate that a mechanism behind the FG puzzle is the same in economies both with Taylor rules and a pre-specified interest-rate path: the smallest eigenvalue is a sufficient statistic for fully characterizing the model's ingredients that generate the backward explosion. The theorems we establish make it possible to see up-front whether or

[10] The analysis of Woodford (2001) requires some additional assumptions; see Woodford (2003) for a discussion and generalizations.

not a specific parameterization of the new Keynesian model leads to the FG puzzle under both ELB and no-ELB scenarios.

Third, we offer a resolution to the FG puzzle. We propose to restrict attention to policy rules that approximate the Ramsey solution. The key principle of the optimal monetary policy is the economy stabilization. Whether the Taylor rules accomplish this goal or not depends on how such rules are parameterized. The parameterizations that satisfy the Taylor Principle and provide sufficient stabilization do not lead to the FG puzzle. In turn, excessively passive Taylor rules that generate the FG puzzle contradict the key stabilization principle of the optimal monetary policy; they are unlikely to represent what the actual monetary authorities have in mind. It is not surprising that in the absence of stabilization, the economy explodes. Thus, to resolve the FG puzzle, we need to modify the way we model the policies rules!

Finally, we help inform central bank decisions related to the FG puzzle. We demonstrate that the central bank that follows a sufficiently active Taylor rule will undo the effect of future shocks, which eliminates the backward-explosion characteristic of the FG puzzle. But then why would the central bank announce a shock in the first place just to undo it with its future actions? Our results suggest FG has a limited usefulness as a stimulating tool if the systematic monetary policy rules are sufficiently active, especially in normal times when ELB does not bind.

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Appendix A. Proofs of Theorem 1 and Corollary 1

In this section, we prove Theorem 1 and its Corollary 1 that establish the regions for the parameters $\phi_\pi \geq 0$, $\phi_{E\pi} \geq 0$ and $\phi_y \geq 0$ corresponding to different types of characteristic roots in the model (1), (2) and (3).

Proof to Theorem 1. The roots to the characteristic equation $m^2 + bm + c\pi_t = 0$ are given by

$$m_1 = \frac{-b + \sqrt{b^2 - 4c}}{2}, \tag{14}$$

$$m_2 = \frac{-b - \sqrt{b^2 - 4c}}{2}, \tag{15}$$

where $b \equiv -1 - \frac{1}{\beta} - \sigma\phi_y - \frac{\sigma\kappa(1-\phi_{E\pi})}{\beta}$, and $c \equiv \frac{(1+\sigma\phi_y)}{\beta} + \frac{\sigma\kappa\phi_\pi}{\beta}$. It is useful to note that $c > 1$.

We start by showing statement ii) of Theorem 1.

Two unstable real roots. To have $|m_1| \geq 1$, $|m_2| \geq 1$, we must have one of the following cases:

$$a) \begin{bmatrix} m_1 \leq -1 \\ m_2 \geq 1 \end{bmatrix}, \quad b) \begin{bmatrix} m_1 \geq 1 \\ m_2 \leq -1 \end{bmatrix}, \quad c) \begin{bmatrix} m_1 \geq 1 \\ m_2 \geq 1 \end{bmatrix}, \quad d) \begin{bmatrix} m_1 \leq -1 \\ m_2 \leq -1 \end{bmatrix}.$$

Let us first rule out *Cases a) and b)* by showing that there are no parameters values that satisfy both restrictions.

Case a) By construction, we have $m_2 < m_1$, so this case is impossible.

Case b) Since $m_1 \geq 1$, we have $\sqrt{b^2 - 4c} \geq 2 + b$.

Since $m_2 \leq -1$, we have $-\sqrt{b^2 - 4c} \leq -2 + b$ which is equivalent $\sqrt{b^2 - 4c} \geq 2 - b$.

i) If $b \geq 0$, then $\sqrt{b^2 - 4c} \geq 2 + b$ implies $\sqrt{b^2 - 4c} \geq 2 - b$, so we only need to insure the former inequality:

$$b^2 - 4c \geq (2 + b)^2 \Rightarrow -4c \geq 4 + 4b, \text{ impossible since } c > 0.$$

ii) If $b < 0$, then $\sqrt{b^2 - 4c} > 2 - b$ implies $\sqrt{b^2 - 4c} > 2 + b$, so again, we only need to insure the former inequality:

$$b^2 - 4c > (2 - b)^2 \Rightarrow -4c > 4 - 4b, \text{ impossible since } c > 0.$$

By combining i) and ii), we conclude that the case b) is impossible.

Case c) Since $m_2 \geq 1$ implies $m_1 \geq 1$, we only need to insure $m_2 \geq 1$, i.e., $\frac{-b-\sqrt{b^2-4c}}{2} \geq 1$. This implies

$$-\sqrt{b^2 - 4c} \geq 2 + b. \tag{16}$$

Since the root is real, we must have $b^2 - 4c \geq 0$. This implies two possibilities: if $b > 0$, we must have $b > 2\sqrt{c}$ and if $b < 0$, we must have $-b > 2\sqrt{c}$. However, the former possibility violates (16), so we are left with $b \leq -2\sqrt{c}$, which leads to boundary value $\phi_{E\pi}^2$:

$$\left(-\frac{1}{\beta} - (1 + \sigma\phi_y) - \frac{\sigma\kappa(1 - \phi_{E\pi})}{\beta}\right) \leq -2\sqrt{\frac{(1 + \sigma\phi_y)}{\beta} + \frac{\sigma\kappa\phi_\pi}{\beta}}$$

$$\phi_{E\pi} \leq 1 + \frac{1}{\sigma\kappa}\left[1 + (1 + \sigma\phi_y)\beta - 2\sqrt{(1 + \sigma\phi_y + \sigma\kappa\phi_\pi)\beta}\right]$$

$$\phi_{E\pi} \leq 1 - \phi_\pi - \frac{(1-\beta)\phi_y}{\kappa} + \frac{1}{\sigma\kappa}\left[1 + (1 + \sigma\phi_y)\beta + \phi_\pi + \frac{(1-\beta)\phi_y}{\kappa} - 2\sqrt{(1 + \sigma\phi_y + \sigma\kappa\phi_\pi)\beta}\right]$$

$$\phi_{E\pi} \leq 1 - \phi_\pi - \frac{(1-\beta)\phi_y}{\kappa} + \frac{1}{\sigma\kappa}\left[1 + \sigma\kappa\phi_\pi + \sigma\phi_y + \beta - 2\sqrt{(1 + \sigma\phi_y + \sigma\kappa\phi_\pi)\beta}\right] \equiv \phi_{E\pi}^2. \tag{17}$$

Furthermore, we re-write (16) as

$$\begin{aligned}\sqrt{b^2-4c} &\le -2-b,\\ \left(\sqrt{b^2-4c}\right)^2 &\le (-2-b)^2,\\ -4c &\le 4+4b.\end{aligned}$$

The latter inequality implies $c+1 \ge -b$, which leads to the boundary value $\phi_{E\pi}^1$:

$$\begin{gathered}1+\frac{\left(1+\sigma\phi_y\right)}{\beta}+\frac{\sigma\kappa\phi_\pi}{\beta} \ge 1+\frac{1}{\beta}+\sigma\phi_y+\frac{\sigma\kappa\left(1-\phi_{E\pi}\right)}{\beta}\\ \phi_{E\pi}^1 \equiv 1-\phi_\pi+\frac{\beta\phi_y}{\kappa}\left(1-\frac{1}{\beta}\right) \le \phi_{E\pi}.\end{gathered}$$

Finally, we consider *Case d)*. The analysis of this case is similar to *Case c)*. Since $m_1 \le -1$ implies $m_2 \le -1$, we only need to insure $m_1 \le -1$, i.e., $\frac{-b+\sqrt{b^2-4c}}{2} \le -1$. This implies

$$\sqrt{b^2-4c} \le -2+b. \tag{18}$$

Since the root is real, we must have $b^2-4c \ge 0$. This implies two possibilities: if $b>0$, we must have $b>2\sqrt{c}$ and if $b<0$, we must have $-b>2\sqrt{c}$. However, the latter possibility violates (18), so we are left with $b \ge 2\sqrt{c}$, which leads to the boundary value $\phi_{E\pi}^3$:

$$\begin{gathered}\left(-\frac{1}{\beta}-\left(1+\sigma\phi_y\right)-\frac{\sigma\kappa\left(1-\phi_{E\pi}\right)}{\beta}\right) \ge 2\sqrt{\frac{\left(1+\sigma\phi_y\right)}{\beta}+\frac{\sigma\kappa\phi_\pi}{\beta}}\\ \phi_{E\pi} \ge 1+\frac{1}{\sigma\kappa}\left[1+\left(1+\sigma\phi_y\right)\beta+2\sqrt{\left(1+\sigma\phi_y+\sigma\kappa\phi_\pi\right)\beta}\right]\\ \phi_{E\pi} \ge 1-\phi_\pi-\frac{(1-\beta)\,\phi_y}{\kappa}+\frac{1}{\sigma\kappa}\left[1+\left(1+\sigma\phi_y\right)\beta+\phi_\pi+\frac{(1-\beta)\,\phi_y}{\kappa}+2\sqrt{\left(1+\sigma\phi_y+\sigma\kappa\phi_\pi\right)\beta}\right]\\ \phi_{E\pi} \ge 1-\phi_\pi-\frac{(1-\beta)\,\phi_y}{\kappa}+\frac{1}{\sigma\kappa}\left[1+\sigma\kappa\phi_\pi+\sigma\phi_y+\beta+2\sqrt{\left(1+\sigma\phi_y+\sigma\kappa\phi_\pi\right)\beta}\right] \equiv \phi_{E\pi}^3.\end{gathered} \tag{19}$$

Furthermore, we re-write (18) as

$$\begin{aligned}\left(\sqrt{b^2-4c}\right)^2 &\le (-2+b)^2\\ -4c &\le 4-4b.\end{aligned}$$

This implies $c+1 \ge b$, which leads to the boundary value $\phi_{E\pi}^4$:

$$\begin{aligned}\frac{\left(1+\sigma\phi_y\right)}{\beta}+\frac{\sigma\kappa\phi_\pi}{\beta}+1 &\ge -1-\frac{1}{\beta}-\sigma\phi_y-\frac{\sigma\kappa\left(1-\phi_{E\pi}\right)}{\beta}\\ \phi_{E\pi} &\le \frac{\left(1+\sigma\phi_y\right)+\sigma\kappa\phi_\pi+\beta+\beta+1+\beta\sigma\phi_y+\sigma\kappa}{\sigma\kappa}\\ \phi_{E\pi} &\le 1-\phi_\pi-\frac{(1-\beta)\,\phi_y}{\kappa}\\ &\quad+\frac{\left(1+\sigma\phi_y\right)+\sigma\kappa\phi_\pi+\beta+\beta+1+\beta\sigma\phi_y+\sigma\kappa}{\sigma\kappa}-1+\phi_\pi+\frac{(1-\beta)\,\phi_y}{\kappa}\\ \phi_{E\pi} &\le 1-\phi_\pi-\frac{(1-\beta)\,\phi_y}{\kappa}+\frac{2\left(1+\sigma\phi_y+\sigma\kappa\phi_\pi+\beta\right)}{\sigma\kappa} \equiv \phi_{E\pi}^4.\end{aligned}$$

We next show statement iii) of Theorem 1.

Two repeated real roots. To have repeated real roots, it must be that $b^2 - 4c = 0$. There are two possible solutions $b = 2\sqrt{c}$ and $b = -2\sqrt{c}$. By using the results (17) and (19) obtained for *Cases c) and d)* of the statement ii), we obtain that the corresponding parameterizations are $\phi_{E\pi} = \phi^2_{E\pi}$ and $\phi_{E\pi} = \phi^3_{E\pi}$.

To see that the resulting root $m = -\frac{b}{2}$ is unstable, notice that $b = 2\sqrt{c}$ and $b = -2\sqrt{c}$ imply $m = -\sqrt{c}$ and $m = \sqrt{c}$, respectively. Since $c > 1$, we conclude that $|m| > 1$.

We now show statement iv) of Theorem 1.

Complex roots. For complex roots, we must have $b^2 - 4c < 0$, which implies $-2\sqrt{c} < b < 2\sqrt{c}$. Again, based on the results (17) and (19) obtained for *Cases c) and d)* of the statement ii), we obtain that the corresponding parameter range is $\phi^2_{E\pi} < \phi_{E\pi} < \phi^3_{E\pi}$. To see that the complex root $m_{1,2} = \mu \pm \eta\iota$ is unstable, we compute $r \equiv \sqrt{\mu^2 + \eta^2} = \sqrt{\left(\frac{-b}{2}\right)^2 + \left(\frac{\sqrt{4c-b^2}}{2}\right)^2} = \sqrt{c} > 1$.

We finally show statement i) of Theorem 1.

One stable and one unstable real roots. To analyze this case, we actually show that there are no parameter values for which we have two stable roots, i.e., $|m_1| < 1$ and $|m_2| < 1$. Indeed, the existence of two stable roots implies that $-1 < m_1 < 1$ and $-1 < m_2 < 1$, i.e.,

$$\begin{aligned} -1 &< \frac{-b + \sqrt{b^2 - 4c}}{2} < 1, \\ -1 &< \frac{-b - \sqrt{b^2 - 4c}}{2} < 1. \end{aligned}$$

If $-1 < \frac{-b-\sqrt{b^2-4c}}{2}$, then we have $-1 < \frac{-b+\sqrt{b^2-4c}}{2}$ and if $\frac{-b+\sqrt{b^2-4c}}{2} < 1$, then we have $\frac{-b-\sqrt{b^2-4c}}{2} < 1$. So, we must check only the following two conditions:

$$\begin{aligned} -1 < \frac{-b - \sqrt{b^2 - 4c}}{2}, \\ \frac{-b + \sqrt{b^2 - 4c}}{2} < 1. \end{aligned}$$

These conditions can be, respectively, re-written as

$$\sqrt{b^2 - 4c} < 2 - b, \tag{20}$$

$$\sqrt{b^2 - 4c} < 2 + b. \tag{21}$$

Since the roots are real, we have $b^2 > 4c$ which means that either $b > \sqrt{4c}$ or $b < -\sqrt{4c}$. Since $c > 1$, these last two inequalities imply that either $b > 2$ or $b < -2$. But then the restrictions (20) and (21) cannot be satisfied simultaneously: if $b > 2$, the right side of (20) is negative and if $b < -2$, the right side of (21) is negative. Since the roots are real and we discarded the possibility of two stable roots, we conclude that we must have one stable and one unstable root, except of those cases when two roots are unstable and when the roots are complex, i.e., everywhere except of the range $\phi^1_{E\pi} \le \phi_{E\pi} \le \phi^4_{E\pi}$. This completes the proof of the statement i) of Theorem 1. ■

Proof to Corollary 1. The roots to the characteristic equation $m^2 + bm + c\pi_t = 0$ are again given by (14) and (15), where under the assumption $\phi_{E\pi} = 0$, we have $b \equiv -1 - \frac{1}{\beta} - \sigma\phi_y - \frac{\sigma\kappa}{\beta}$, and $c \equiv \frac{(1+\sigma\phi_y)}{\beta} + \frac{\sigma\kappa\phi_\pi}{\beta}$. It is useful to note that $b < 0$ and $c > 1$.

We start by showing statement ii) of Corollary 1.

Two unstable real roots. To have $|m_1| \geq 1$, $|m_2| \geq 1$, we must have one of the following cases:

$$a) \begin{bmatrix} m_1 \leq -1 \\ m_2 \geq 1 \end{bmatrix}, \qquad b) \begin{bmatrix} m_1 \geq 1 \\ m_2 \leq -1 \end{bmatrix}, \qquad c) \begin{bmatrix} m_1 \geq 1 \\ m_2 \geq 1 \end{bmatrix}, \qquad d) \begin{bmatrix} m_1 \leq -1 \\ m_2 \leq -1 \end{bmatrix}.$$

Cases a) and b) The results of Theorem 1 apply here as well, so we conclude that these two cases are impossible.

Case c) Since $m_2 \geq 1$ implies $m_1 \geq 1$, we only need to insure $m_2 \geq 1$, i.e., $\frac{-b-\sqrt{b^2-4c}}{2} \geq 1$. Like in *Case c)* of Theorem 1, this implies that $b \leq -2\sqrt{c}$ and results in the corresponding boundary value ϕ_π^2:

$$\begin{aligned} \left(-\frac{1}{\beta} - \left(1+\sigma\phi_y\right) - \frac{\sigma\kappa}{\beta}\right) &\leq -2\sqrt{\frac{\left(1+\sigma\phi_y\right)}{\beta} + \frac{\sigma\kappa\phi_\pi}{\beta}} \\ \frac{1}{\beta} + \left(1+\sigma\phi_y\right) + \frac{\sigma\kappa}{\beta} &\geq 2\sqrt{\frac{\left(1+\sigma\phi_y\right)}{\beta} + \frac{\sigma\kappa\phi_\pi}{\beta}} \\ \phi_\pi &\leq 1 - \frac{\left(1-\beta\right)\phi_y}{\kappa} + \frac{\beta}{4\sigma\kappa}\left\{1 - \frac{1}{\beta} - \sigma\phi_y - \frac{\sigma\kappa}{\beta}\right\}^2 \equiv \phi_\pi^2. \end{aligned} \tag{22}$$

The boundary value ϕ_π^1 follows by (16). Like in Case 3 of Theorem 1, we have $c+1 \geq -b$ and consequently, we obtain

$$\begin{aligned} 1 + \frac{\left(1+\sigma\phi_y\right)}{\beta} + \frac{\sigma\kappa\phi_\pi}{\beta} &\geq 1 + \frac{1}{\beta} + \sigma\phi_y + \frac{\sigma\kappa}{\beta} \\ \phi_\pi^1 \equiv 1 - \frac{\left(1-\beta\right)\phi_y}{\kappa} &\leq \phi_\pi. \end{aligned} \tag{23}$$

Finally, we consider *Case d)*. Following the reasoning of the corresponding case of Theorem 1, we conjecture that we must have $b \geq 2\sqrt{c}$. But this is not possible since by definition, we have $b < 0$ and $c > 1$. Thus, unlike Theorem 1, here we do not have boundary values analogous to $\phi_{E\pi}^3$ and $\phi_{E\pi}^4$.

We next show statement iii) of Corollary 1.

Two repeated real roots. To have repeated real roots, it must be that $b^2 - 4c = 0$. There are two possible solutions $b = 2\sqrt{c}$ and $b = -2\sqrt{c}$. But in the present case, we have $b < 0$, only the latter root is possible. Using the results (22) obtained for *Case c)* of the statement ii), we obtain that the corresponding parameterization is $\phi_\pi = \phi_\pi^2$. Again, to see that the resulting root $m = -\frac{b}{2}$ is unstable, notice that $b = -2\sqrt{c}$ imply $m = \sqrt{c}$, respectively. Since $c > 1$, we conclude that $|m| > 1$.

We now by show statement iv) of Corollary 1.

Complex roots. For complex roots, we must have $b^2 - 4c < 0$, which implies $-2\sqrt{c} < b < 2\sqrt{c}$. The argument of *Case d)* of the present proof rules out the possibility of $b \geq 2\sqrt{c}$ so that the second inequality always holds. Therefore, the roots are complex whenever $-2\sqrt{c} < b$, which together with the result (22) obtained for *Case c)* of the statement ii) leads us to $\phi_\pi > \phi_\pi^2$. Like in Theorem 1, the complex root $m_{1,2} = \mu \pm \eta\iota$ is unstable since $r \equiv \sqrt{\mu^2 + \eta^2} = \sqrt{\left(\frac{-b}{2}\right)^2 + \left(\frac{\sqrt{4c-b^2}}{2}\right)^2} = \sqrt{c} > 1$.

We finally show statement i) of Theorem 1.

One stable and one unstable real roots. The arguments of Theorem 1 apply to this case as well as, so that we conclude that there are no parameter values for which we have two stable roots, i.e., $|m_1| < 1$ and $|m_2| < 1$. Since the roots are real and we discarded the possibility of two stable roots, we conclude that we must have one stable and one unstable root, except when two roots are unstable and when the roots are complex which yields the range $\phi_\pi < \phi_\pi^1$ and completes the proof of the statement. ■

Appendix B. Method of undetermined coefficients for linear stochastic models

In the economy with stochastic shocks, we should construct conditional expectations for future shocks. Our closed-form solutions provide a convenient way of modeling a variety of uncertainty scenarios, including temporary and permanent shocks, anticipated and unanticipated shocks, as well as mixtures of deterministic trends and stochastic shocks. Since the characteristic roots in closed-form solutions of Theorem 2 are non-random, the expectation operator can be brought inside the summations, for example, $E_t\left[\sum_{s=t}^{\infty} m_1^{t-1-s} z_s\right] = \sum_{s=t}^{\infty} m_1^{t-1-s} E_t[z_s]$, so that effectively, we need to compute $E_t[z_s]$ for $s \geq t$.

As an illustration, let us assume that z_t follows a first-order autoregressive process $z_{t+1} = \rho z_t + \varepsilon_{t+1}$, in which case, we have $E_t[z_s] = \rho^{s-t} z_t$ for $s \geq t$. Furthermore, let us assume that the roots are complex, i.e., case iv). Then, the complex-root solution in Theorem 2 can be re-written as

$$\pi_t = C_1 r^t \cos(\theta t) + C_2 r^t \sin(\theta t) + \frac{z_t}{\eta\rho}\left[\sum_{s=t}^{\infty}\left(\frac{r}{\rho}\right)^{t-1-s} \sin(\theta(t-1-s))\right]. \tag{24}$$

To simulate stochastic time-series solution, we draw a sequence of shocks for z_t, find π_t from (24) and compute x_t from (2). Similar formulas are easy to show for the remaining cases; in those cases, the roots m_i can be adjusted to ρ by $\frac{m_i}{\rho}$ and the term $\frac{z_t}{\rho}$ can be taken out of the summation. Examples of time-series solutions to the stochastic versions of the model are shown in Appendices D and E.

To ensure stationarity in the stochastic case, we need to impose the same restrictions on the homogeneous solutions as those necessary for forward stability in the deterministic case. In particular, we obtain a unique stationary solution in cases ii)–iv) by setting $C_1 = 0$ and $C_2 = 0$, and there is a multiplicity of stationary solutions in case i) since any C_2 is consistent with stationarity.

Closed-form solutions to new Keynesian models with uncertainty are studied in Taylor (1986) by using a method of undetermined coefficients – an analytical technique that reduces a stochastic difference equation to the deterministic one. In contrast, our present analysis proceeds in the opposite direction: we first constructed a solution to the deterministic model, and we then generalized it to the stochastic case. Below, we show that both approaches lead to the same stochastic solution under a general linear process for shock z_t. Taylor's (1986) method does not specify how to construct a solution to the deterministic model like those we obtain in Theorem 2, which is our main contribution. Finally, Cochrane (2017b) constructs related solutions for the stochastic version of the model in which one root is stable and the other root us unstable, which corresponds to our case i).

We assume that z_t follows a general linear process with a representation

$$z_s = \sum_{j=0}^{\infty} \vartheta_j \varepsilon_{s-j}, \tag{25}$$

where ϑ_j is a sequence of parameters, and ε_t is a serially uncorrelated random variable with zero mean. The process (25) includes important types of policy shocks as special cases, in particular, it allows us to distinguish between temporary and permanent shocks, as well as anticipated and unanticipated shocks; see Taylor (1986) for a discussion. We first construct a closed-form solution of our Theorem 2 under (25), and we next show that the method of undetermined coefficients of Taylor (1986) leads to the same solution. We omit the homogeneous solution because it is the same in the deterministic and stochastic models, and we concentrate on particular solutions.

Closed-form solutions of Theorem 2. As an example, consider the closed-form solution (**??**) of Theorem 2 for the model with complex roots, which under assumption (25) becomes

$$\pi_t = \frac{1}{\eta} E_t\left[\sum_{s=t}^{\infty} h_{t-1-s} \sum_{j=0}^{\infty} \theta_j \varepsilon_{s-j}\right],$$

where $h_{t-1-s} \equiv r^{t-1-s} \sin(\theta(t-1-s))$ is compact notation. The latter expression can be written as

$$\pi_t = \frac{1}{\eta} E_t \left[h_{-1} \sum_{j=0}^{\infty} \theta_j \varepsilon_{t-j} + h_0 \sum_{j=1}^{\infty} \theta_j \varepsilon_{t+1-j} + h_1 \sum_{j=0}^{\infty} \theta_j \varepsilon_{t+2-j} + ... \right].$$

Since $E_t[\varepsilon_{t+\tau-j}] = 0$ for any $\tau \geq 0$, we can compute expectation with the following sequence of steps:

$$\begin{aligned} \pi_t = \frac{1}{\eta} \left[h_{-1} \sum_{j=0}^{\infty} \theta_j \varepsilon_{t-j} + h_0 \sum_{j=1}^{\infty} \theta_j \varepsilon_{t+1-j} + h_1 \sum_{j=2}^{\infty} \theta_j \varepsilon_{t+2-j} + ... \right] \\ = \frac{1}{\eta} \left[h_{-1} \sum_{j=0}^{\infty} \theta_j \varepsilon_{t-j} + h_0 \sum_{j=0}^{\infty} \theta_{j+1} \varepsilon_{t-j} + h_1 \sum_{j=0}^{\infty} \theta_{j+2} \varepsilon_{t-j} + ... \right] \\ = \frac{1}{\eta} \sum_{j=0}^{\infty} \varepsilon_{t-j} \left[h_{-1}\theta_j + h_0 \theta_{j+1} + h_1 \theta_{j+2} + ... \right] = \frac{1}{\eta} \sum_{j=0}^{\infty} \varepsilon_{t-j} \sum_{s=j}^{\infty} h_{j-1-s} \theta_s. \end{aligned} \quad (26)$$

Method of undetermined coefficients. The method of undetermined coefficients described in Taylor (1986) requires us to guess that a solution for π_t has the same kind of representation as (25), specifically,

$$\pi_t = \sum_{j=0}^{\infty} \varepsilon_{t-j} \gamma_j, \quad (27)$$

where γ_j is a sequence of unknown coefficients. By taking into account that $E_t[\gamma_0 \varepsilon_{t+1}] = 0$, we obtain $E_t[\pi_{t+1}] = \sum_{j=1}^{\infty} \gamma_j \varepsilon_{t-j+1}$ and $E_{t+1}[\pi_{t+2}] = \sum_{j=2}^{\infty} \gamma_j \varepsilon_{t-j+2}$. Therefore, we can re-write (4) as

$$\sum_{j=2}^{\infty} \gamma_j \varepsilon_{t-j+2} + b \sum_{j=1}^{\infty} \gamma_j \varepsilon_{t-j+1} + c \sum_{j=0}^{\infty} \gamma_j \varepsilon_{t-j} = - \sum_{j=0}^{\infty} \vartheta_j \varepsilon_{t-j}. \quad (28)$$

Equating the coefficients on both sides of the equality (28) gives us a set of restrictions

$$\gamma_{j+2} + b\gamma_{j+1} + c\gamma_j = -\vartheta_j, \qquad j = 0, 1, ... \quad (29)$$

This is a deterministic difference equation with a forcing variable ϑ_j. It has the same structure as a stochastic difference equation and it is identical up to notation to the deterministic version of the equation (4) studied in the main text. Therefore, the coefficients of the stochastic equation (29) can be described by closed-form solutions of Theorem 2, again up to notation. For example, Theorem 2, case iv) implies that the sequence of the coefficients (29) in the model with complex roots is given by $\gamma_j = \frac{1}{\eta} \left[\sum_{s=j}^{\infty} h_{j-1-s} \vartheta_s \right]$, which together with (27) implies the same solution for π_t as (26). The equivalence for the remaining cases i)-iii) can be shown similarly.

Appendix C. Supplementary results for Section 3

In this section, we present some supplementary results for Section 3.

C1. Taylor rule with actual inflation and output gap: the case of the real roots

In Section 3.3, we compare the solution under two Taylor rules (3) with output gap: one contains the actual inflation and the other contains the expected inflation. In the former case, the model is parameterized by $\phi_y = 0.5$ and $\phi_{E\pi} = 2$, and in the latter case, it is parameterized by $\phi_y = 0.5$ and $\phi_\pi = 2$. These parameterizations lead to complex roots, which correspond to case iv) of Theorems 1-3.

Interestingly, a relatively small change in parameterization produces a switch to the real roots, which is the case ii) or iii) of Theorems 1, 2 and Corollary 1. As an example, we show the solutions for slightly different parameterizations, namely, $\phi_y = 0.5$ and $\phi_{E\pi} = 1.5$ and $\phi_y = 0.5$ and $\phi_\pi = 1.5$.

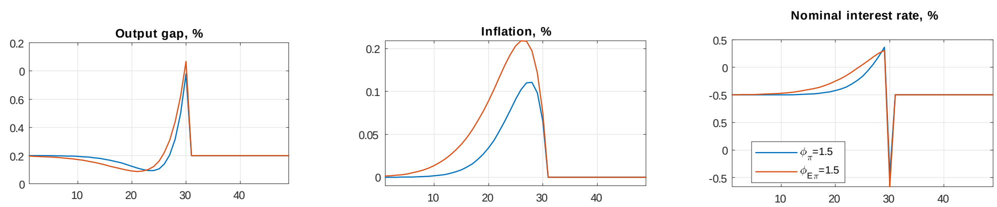

Figure C1. Taylor rule with output gap $\phi_y = 0.5$ and expected inflation $\phi_{E\pi} = 1.5$ vesus actual inflation $\phi_\pi = 1.5$.

Qualitatively, the solutions shown in Figure C1 are very similar to those reported in Figure 3 in the main text. Quantitatively, the difference in inflation between the two solutions is somewhat larger under $\phi_{E\pi} = 1.5$ and $\phi_\pi = 1.5$ than under $\phi_{E\pi} = 2$ and $\phi_\pi = 2$ reported in the main text.

C2. Active Taylor rule with expected inflation

To study the robustness of the FG puzzle, let us consider a more general Taylor rule (3) with expected inflation such that $\phi_{E\pi} > 1$; for example, in Figure C.2, we illustrate the case $\phi_{E\pi} = 3$, $\phi_\pi = 0$ and $\phi_y = 0$. Here, we are in case iv) of Theorems 2 with unstable complex roots. In response to the shock in period 30, the model's variables start fluctuating from period 0 and on, up to period 30.

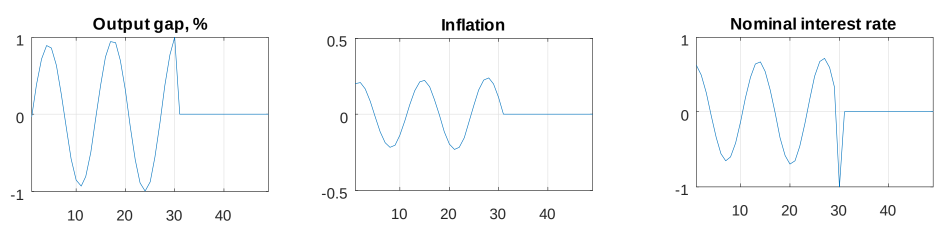

Figure C.2. Taylor rule with expected inflation, $\phi_{E\pi} = 3$.

There is a gradual decay but it is very slow, since we are close to unit root $|r| \approx 1.005$ under our benchmark calibration. While this oscillating case is discomforting, we draw attention that it is produced by a meaningful positive (although too strong) response of the interest rate to expected inflation; we cannot rule it out as a less appealing case of negative Taylor-rule coefficients, also leading to oscillations.

C3. Active Taylor rule with actual inflation

In Figure C3, we show the solution under the Taylor rule (3) with $\phi_{E\pi} = 0$, $\phi_\pi = 3$ and $\phi_y = 0$.

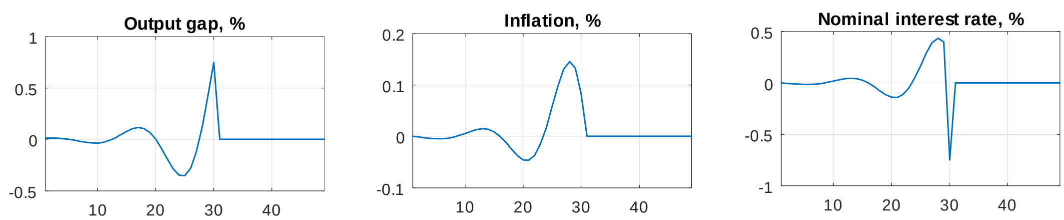

Figure C3. Taylor rule with actual inflation, $\phi_\pi = 3$.

We still observe oscillations that result from the complex roots but now the decay of all variables is much faster than in Figure C2.

Appendix D. The basic nonlinear new Keynesian model

In this section, we describe the basic new Keynesian model that leads to the three-equation model (1), (2) and (3) studied in the main text. Also, we present some robustness results.

D1. The model

Households. The representative household solves

$$\max_{\{C_t,L_t,B_t\}_{t=0,\ldots,\infty}} E_0 \sum_{t=0}^{\infty} \beta^t \exp\left(\eta_{u,t}\right) \left[\frac{C_t^{1-\sigma}-1}{1-\sigma} - \exp\left(\eta_{L,t}\right) \frac{L_t^{1+\vartheta}-1}{1+\vartheta} \right] \tag{30}$$

$$\text{s.t. } P_t C_t + \frac{B_t}{\exp\left(\eta_{B,t}\right) R_t} + T_t = B_{t-1} + W_t L_t + \Pi_t, \tag{31}$$

where $\left(B_0, \eta_{u,0}, \eta_{L,0}, \eta_{B,0}\right)$ is given; C_t, L_t, and B_t are consumption, labor and nominal bond holdings, respectively; P_t, W_t and R_t are the commodity price, nominal wage and (gross) nominal interest rate, respectively; $\eta_{u,t}$ and $\eta_{L,t}$ are exogenous preference shocks to the overall momentary utility and disutility of labor, respectively; $\eta_{B,t}$ is an exogenous premium in the return to bonds; T_t is lump-sum taxes; Π_t is the profit of intermediate-good firms; $\beta \in (0,1)$ is the discount factor; $\sigma > 0$ and $\vartheta > 0$ are the utility-function parameters. The shocks follow standard AR(1) processes with means zero and constant standard deviations.

Final-good firms. Perfectly competitive final-good firms produce final goods using intermediate goods. A final-good firm buys $Y_t(i)$ of an intermediate good $i \in [0,1]$ at price $P_t(i)$ and sells Y_t of the final good at price P_t in a perfectly competitive market. The profit-maximization problem is

$$\max_{Y_t(i)} \quad P_t Y_t - \int_0^1 P_t(i)\, Y_t(i)\, di \tag{32}$$

$$\text{s.t. } Y_t = \left(\int_0^1 Y_t(i)^{\frac{\varepsilon-1}{\varepsilon}} di \right)^{\frac{\varepsilon}{\varepsilon-1}}, \tag{33}$$

where (33) is a Dixit-Stiglitz aggregator function with $\varepsilon \geq 1$.

Intermediate-good firms. Monopolistic intermediate-good firms produce intermediate goods using labor and are subject to sticky prices. The firm i produces the intermediate good i. To choose labor in each period t, the firm i minimizes the nominal total cost, TC (net of government subsidy v),

$$\min_{L_t(i)} \quad \mathrm{TC}\left(Y_t(i)\right) = (1-v)\, W_t L_t(i) \tag{34}$$

$$\text{s.t. } Y_t(i) = \exp\left(\eta_{a,t}\right) L_t(i), \tag{35}$$

$$\eta_{a,t+1} = \rho_a \eta_{a,t} + \epsilon_{a,t+1}, \qquad \epsilon_{a,t+1} \sim \mathcal{N}\left(0, \sigma_a^2\right), \tag{36}$$

where $L_t(i)$ is the labor input; $\exp\left(\eta_{a,t}\right)$ is the productivity level such that $\eta_{a,t}$ follows the standard AR(1) process. The firms are subject to Calvo-type price setting: a fraction $1-\theta$ of the firms sets prices optimally, $P_t(i) = \widetilde{P}_t$, for $i \in [0,1]$, and the fraction θ is not allowed to change the price and maintains the same price as in the previous period, $P_t(i) = P_{t-1}(i)$, for $i \in [0,1]$. A reoptimizing firm $i \in [0,1]$ maximizes the current value of profit over the time when $\widetilde{P}_t$ remains effective,

$$\max_{\widetilde{P}_t} \quad \sum_{j=0}^{\infty} \beta^j \theta^j E_t \left\{ \Lambda_{t+j} \left[\widetilde{P}_t Y_{t+j}(i) - P_{t+j} \mathrm{mc}_{t+j} Y_{t+j}(i) \right] \right\} \tag{37}$$

$$\text{s.t. } Y_t(i) = Y_t \left(\frac{P_t(i)}{P_t} \right)^{-\varepsilon}, \tag{38}$$

where (38) is the demand for an intermediate good i (follows from the first-order condition of (32), (33)); Λ_{t+j} is the Lagrange multiplier on the household's budget constraint (31); mc_{t+j} is the real marginal cost of output at time $t+j$ (which is identical across the firms).

Government. Government finances a stochastic stream of public consumption by levying lump-sum taxes and by issuing nominal debt. The government budget constraint is

$$T_t + \frac{B_t}{\exp\left(\eta_{B,t}\right) R_t} = P_t \frac{\overline{G} Y_t}{\exp\left(\eta_{G,t}\right)} + B_{t-1} + vW_t L_t, \tag{39}$$

where $\overline{G}$ is the steady-state share of government spending in output; vW_tL_t is the subsidy to the intermediate-good firms; $\eta_{G,t}$ is a government-spending shock, that follows the standard AR(1) process.

Monetary authority. The monetary authority follows a Taylor rule

$$R_t \equiv R_* \left(\frac{R_{t-1}}{R_*}\right)^{\mu} \left[\left(\frac{E_t \pi_{t+1}}{\pi_{tar}}\right)^{\phi_{E\pi}} \left(\frac{\pi_t}{\pi_{tar}}\right)^{\phi_\pi} \left(\frac{Y_t}{Y_{N,t}}\right)^{\phi_y}\right]^{1-\mu} \exp\left(\eta_{R,t}\right), \tag{40}$$

where R_t is the gross nominal interest rate at t; R_* is the steady state level of nominal interest rate; π_{tar} is the target inflation; $Y_{N,t}$ is the natural level of output; and $\eta_{R,t}$ is a monetary shock following the standard AR(1) process.

Natural level of output. The natural level of output $Y_{N,t}$ is the level of output in an otherwise identical economy but without distortions. It is a solution to the following planner's problem

$$\max_{\{C_t, L_t\}_{t=0,\ldots,\infty}} E_0 \sum_{t=0}^{\infty} \beta^t \exp\left(\eta_{u,t}\right) \left[\frac{C_t^{1-\sigma}-1}{1-\sigma} - \exp\left(\eta_{L,t}\right) \frac{L_t^{1+\vartheta}-1}{1+\vartheta}\right] \tag{41}$$

$$\text{s.t. } C_t = \exp\left(\eta_{a,t}\right) L_t - G_t, \tag{42}$$

where $G_t \equiv \frac{\overline{G} Y_t}{\exp\left(\eta_{G,t}\right)}$ is given, and $\eta_{u,t+1}$, $\eta_{L,t+1}$, $\eta_{a,t+1}$, and $\eta_{G,t}$ follow the same processes as in the non-optimal economy. The FOCs of the problem (41), (42) imply that $Y_{N,t}$ depends only on exogenous shocks (see equation (51) below).

Equilibrium conditions. We summarize the equilibrium conditions below:

$$S_t = \frac{\exp\left(\eta_{u,t}+\eta_{L,t}\right)}{\exp\left(\eta_{a,t}\right)} L_t^{\vartheta} Y_t + \beta\theta E_t\left\{\pi_{t+1}^{\varepsilon} S_{t+1}\right\}, \tag{43}$$

$$F_t = \exp\left(\eta_{u,t}\right) C_t^{-\sigma} Y_t + \beta\theta E_t\left\{\pi_{t+1}^{\varepsilon-1} F_{t+1}\right\}, \tag{44}$$

$$C_t^{-\sigma} = \frac{\beta \exp\left(\eta_{B,t}\right) R_t}{\exp\left(\eta_{u,t}\right)} E_t\left[\frac{C_{t+1}^{-\sigma} \exp\left(\eta_{u,t+1}\right)}{\pi_{t+1}}\right], \tag{45}$$

$$\frac{S_t}{F_t} = \left[\frac{1-\theta\pi_t^{\varepsilon-1}}{1-\theta}\right]^{\frac{1}{1-\varepsilon}}, \tag{46}$$

$$\Delta_t = \left[(1-\theta)\left[\frac{1-\theta\pi_t^{\varepsilon-1}}{1-\theta}\right]^{\frac{\varepsilon}{\varepsilon-1}} + \theta\frac{\pi_t^{\varepsilon}}{\Delta_{t-1}}\right]^{-1}, \tag{47}$$

$$
\begin{aligned}
Y_t &= \exp\left(\eta_{a,t}\right) L_t \Delta_t, && (48)\\
C_t &= \left(1 - \frac{\overline{G}}{\exp\left(\eta_{G,t}\right)}\right) Y_t, && (49)
\end{aligned}
$$

$$
R_t = R_* \left(\frac{R_{t-1}}{R_*}\right)^{\mu} \left[\left(\frac{E_t \pi_{t+1}}{\pi_{tar}}\right)^{\phi_{E\pi}} \left(\frac{\pi_t}{\pi_{tar}}\right)^{\phi_\pi} \left(\frac{Y_t}{Y_{N,t}}\right)^{\phi_y}\right]^{1-\mu} \exp\left(\eta_{R,t}\right), \tag{50}
$$

and $Y_{N,t}$ is given by

$$
Y_{N,t} = \left[\frac{\exp\left(\eta_{a,t}\right)^{(\sigma+\vartheta)(1-\sigma)}}{\left[1 - \frac{\overline{G}}{\exp\left(\eta_{G,t}\right)}\right]^{\sigma} \exp\left(\eta_{L,t}\right)}\right]^{\frac{1}{\vartheta+\sigma}}. \tag{51}
$$

Here, the variables S_t and F_t are introduced for a compact representation of the profit-maximization condition of the intermediate-good firm and are defined recursively; $\pi_{t+1} \equiv \frac{P_{t+1}}{P_t}$ is the gross inflation rate between t and $t+1$; Δ_t is a measure of price dispersion across firms (also referred to as efficiency distortion). To get condition (43), we impose $\frac{\varepsilon}{\varepsilon-1}(1-v) = 1$, which ensures that the model admits a deterministic steady state (this assumption is commonly used in the related literature; see, e.g., Christiano et al. 2005). An interior equilibrium is described by 8 equilibrium conditions (43)–(50), and 6 processes for exogenous shocks. The system of equations must be solved with respect to 8 unknowns $\{C_t, Y_t, L_t, \pi_t, \Delta_t, R_t, S_t, F_t\}$. There are 2 endogenous and 6 exogenous state variables, $\{\Delta_{t-1}, R_{t-1}\}$, and $\left\{\eta_{u,t}, \eta_{L,t}, \eta_{B,t}, \eta_{a,t}, \eta_{R,t}, \eta_{G,t}\right\}$, respectively.

Linearized equilibrium conditions. Below, we provide linearized versions of the equilibrium conditions (43)–(51):

$$
\begin{aligned}
S_t - S_* &= L_*^{\vartheta} Y_* (\eta_{u,t} + \eta_{L,t} - \eta_{a,t}) - \vartheta L_*^{\vartheta-1}(L - L_*) Y_* \\
&\quad + L_*^{\vartheta}(Y_t - Y_*) + \beta\theta S_* \varepsilon \pi_*^{\varepsilon-1} E_t(\pi_{t+1} - \pi_*) + \beta\theta\varepsilon\pi_*^{\varepsilon-1}\pi_*^{\varepsilon} E_t\left(S_{t+1} - S_*\right),
\end{aligned}
$$

$$
\begin{aligned}
F_t - F_* &= \eta_{u,t} C_*^{-\sigma} Y_* + (-\sigma) C_*^{-\sigma-1} Y_* (C_t - C_*) + C_*^{-\sigma}(Y_t - Y_*) \\
&\quad + \beta\theta(\varepsilon - 1)\pi_*^{\varepsilon-2} F_* E_t\left(\pi_{t+1} - \pi_*\right) + \beta\theta\varepsilon\pi_*^{\varepsilon-1} E_t(F_{t+1} - F_*),
\end{aligned}
$$

$$
\begin{aligned}
-\sigma C_*^{-\sigma-1}(C_t - C_*) &= \beta R_* \frac{C_*^{-\sigma}}{\pi_*}(\eta_{B,t} - \eta_{u,t}) + \beta \frac{C_*^{-\sigma}}{\pi_*}(R_t - R_*) \\
&\quad + \beta R_*(-\sigma)\frac{C_*^{-\sigma-1}}{\pi_*} E_t(C_{t+1} - C_*) - \beta R_* \frac{C_*^{-\sigma}}{\pi_*^2} E_t(\pi_{t+1} - \pi_*) \\
&\quad + \beta R_*(-\sigma)\frac{C_*^{-\sigma}}{\pi_*}\rho_{\eta_u}\eta_{u,t},
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{F_*}\left(S_t - S_*\right) + \frac{S_*}{F_*^2}(F_t - F_*) &= \\
&= \left(\frac{1 - \theta\pi_*^{\varepsilon-1}}{(1-\theta)}\right)^{\frac{1}{1-\varepsilon}-1} \frac{\theta}{1-\theta}\pi_*^{\varepsilon-2}(\pi_t - \pi_*),
\end{aligned}
$$

$$
\begin{aligned}
\Delta_t - \Delta_* &= \left[\frac{(1-\theta)\left(1 - \theta\pi_*^{\varepsilon-1}\right)}{1-\theta}\right]^{\frac{\varepsilon}{\varepsilon-1}} + \theta\frac{\pi_*^{\varepsilon}}{\Delta_*^{-2}}\left[\frac{1 - \theta\pi_*^{\varepsilon-1}}{1-\theta}\right]^{\frac{\varepsilon}{\varepsilon-1}-1} \varepsilon\theta\pi_*^{\varepsilon-2} \\
&\quad - \theta\frac{\varepsilon}{\Delta_*}\pi_*^{\varepsilon-1}\left(\pi_t - \pi_*\right) + \theta\frac{\pi_*^{\varepsilon}}{\Delta_*^2}\left(\Delta_{t-1} - \Delta_*\right),
\end{aligned}
$$

$$Y_t - Y_* = \eta_{a,t} L_* \Delta_* + (L_t - L_*)\Delta_* + (\Delta_t - \Delta_*)L_*,$$

$$C_t - C_* = \frac{1-\overline{G}}{Y_*}\eta_{G,t} + (1-\overline{G})(Y_t - Y_*),$$

$$\begin{aligned} R_t - R_* \;=\; & R_* \eta_{R,t} + \mu(R_{t-1} - R_*) + (1-\mu)\phi_\pi \frac{R_*}{\pi_*}(\pi_t - \pi_*) + \\ & (1-\mu)\phi_{E\pi}\frac{R_*}{\pi_*}E_t(\pi_{t+1} - \pi_*) + (1-\mu)\phi_y \frac{R_*}{Y_*}(Y_t - Y_*) \end{aligned}$$

$$\begin{aligned} Y_{N,t} - Y_{N_*} \;=\; & \frac{1}{\vartheta + \sigma}(1-\overline{G})^{\frac{-\sigma}{\vartheta+\sigma}-1}\left[\frac{1+\vartheta}{1-\overline{G}}\right]^\sigma \eta_{a,t} \\ & + \sigma(1-\overline{G})^{-\sigma-1}\eta_{G,t} + \eta_{L,t}(1-\overline{G})^{-\sigma}. \end{aligned}$$

Under the assumptions of no government spendings, no shocks, and no endogenous natural level of output, the above nine linearized equilibrium conditions can be reduced to three equations (1)–(3) used in the main text; see, e.g., Galí (2008).

Calibration procedure. We assume $\sigma = 1$ and $\vartheta = 2.09$ in the utility function (30); $\mu = 0.82$ in the Taylor rule (50); $\varepsilon = 4.45$ in the production function of the final-good firm (33); $\theta = 0.83$ (the fraction of the intermediate-good firms affected by price stickiness); $\overline{G} = 0.23$ in the government budget constraint (39). We set the discount factor at $\beta = 0.99$. To parameterize the Taylor rule (50), we use the steady-state interest rate $R_* = \frac{\pi_*}{\beta}$, and the target inflation, $\pi_* = 1$ (a zero net inflation target). In a stochastic version of the model, we calibrate the parameters in the processes for shocks as follows: In the AR(1) processes for shocks, we assume the autocorrelation coefficients, $\rho_u = 0.92$, $\rho_G = 0.95$, $\rho_L = 0.25$, $\rho_a = 0.95$, $\rho_B = 0.22$, $\rho_R = 0.15$, and the standard deviations of shocks $\sigma_u = 0.054\%$, $\sigma_G = 0.038\%$, $\sigma_L = 0.018\%$, $\sigma_a = 0.045\%$, $\sigma_B = 0.023\%$ and $\sigma_R = 0.028\%$ (these values come from Del Negro et al., 2007, and from Smets and Wouters, 2007).

In the three-equation model, we use similar parameter values, namely, the slope of the Phillips curve $\kappa = \frac{(1-\beta\theta)(1-\theta)}{\theta}(1+\vartheta)$ is computed under the same values of the parameters β, θ, ϑ; the coefficient of relative risk aversion is $\sigma = 1$.

Solution procedure. In Section 6, we solve linear and nonlinear versions of the model by using extended path (EP) method by Fair and Taylor (1983). The model starts in the steady state, in particular, we assume $R_* = \frac{\pi_*}{\beta}$. In the initial period, the monetary authority announces that at $t = 30$, the nominal interest rate will go down by 1%. We then construct the path for the model's variables to satisfy the model's equations. We solve the model for 50 periods, and we extend the path to 150 periods.

Robustness of our findings

Up to now, we have studied a linearized version of the basic new Keynesian model that admits closed-form solutions. More general versions of the model do not admit closed-form solutions, so we resort to numerical analysis. The model with FG is non-stationary, the optimal decision rules change from one period to another, driven by anticipatory effects, and the conventional numerical methods that construct time-invariant value and policy functions are not applicable. We use two methods that are designed to solve such models: an extended path method of Fair and Taylor (1983) and an extended function path method of Maliar, Maliar, Taylor and Tsener (2020). In both methods, we impose (forward) stability: for the former method, we assume that in the terminal period, the economy arrives in the steady state, while for the latter method, we assume that the economy asymptotically converges to stationary. Thus, all the equilibria in our simulations are forward stable equilibria by construction. We first ask whether or not our findings are robust to the introduction of nonlinearity; we then introduce uncertainty in the nonlinear model by assuming six exogenous shocks; and we finally augment the nonlinear model to include capital.

7.0.1 Nonlinearity

In this section, we consider a nonlinear version of the basic new Keynesian model. The difference between global and local determinacy might show up in the behavior of the real economy in times of extreme inflation (not just in ZLB). Specifically, we consider a nonlinear new Keynesian model analyzed in Maliar and Maliar (2015). The economy is populated by households, final-good firms, intermediate-good firms, monetary authority and government. In particular, the monetary authority follows a Taylor rule

$$R_t \equiv R_* \left(\frac{R_{t-1}}{R_*}\right)^{\mu} \left[\left(\frac{E_t \pi_{t+1}}{\pi_{tar}}\right)^{\phi_{E\pi}} \left(\frac{\pi_t}{\pi_{tar}}\right)^{\phi_\pi} \left(\frac{Y_t}{Y_{N,t}}\right)^{\phi_y}\right]^{1-\mu} \exp\left(\eta_{R,t}\right), \tag{52}$$

where R_t is the gross nominal interest rate at t; R_* is the steady state level of nominal interest rate; π_{tar} is the target inflation; $Y_{N,t}$ is the natural level of output; and $\eta_{R,t}$ is a monetary shock following the standard first-order autoregressive process. In addition to the three variables in our baseline linearized model of Section 2, the nonlinear model has consumption, labor, price dispersion and the supplementary variables S and F following from the profit maximizing conditions of monopolistic firms; see Appendix A for the model's description, the list of the first-order, the calibration and solution procedures, as well as a list of linearized equations. [11]

As an example, in Figure D1, we compare the effect of FG on linear and nonlinear solutions under the Taylor rule (52) parameterized by $\phi_{E\pi} = 0$, $\phi_\pi = 1/\beta$ and $\phi_y = 0$ (we assume $\mu = 0$). The anticipated shock here (as well as in the rest of Section 5) happens at $T = 20$.

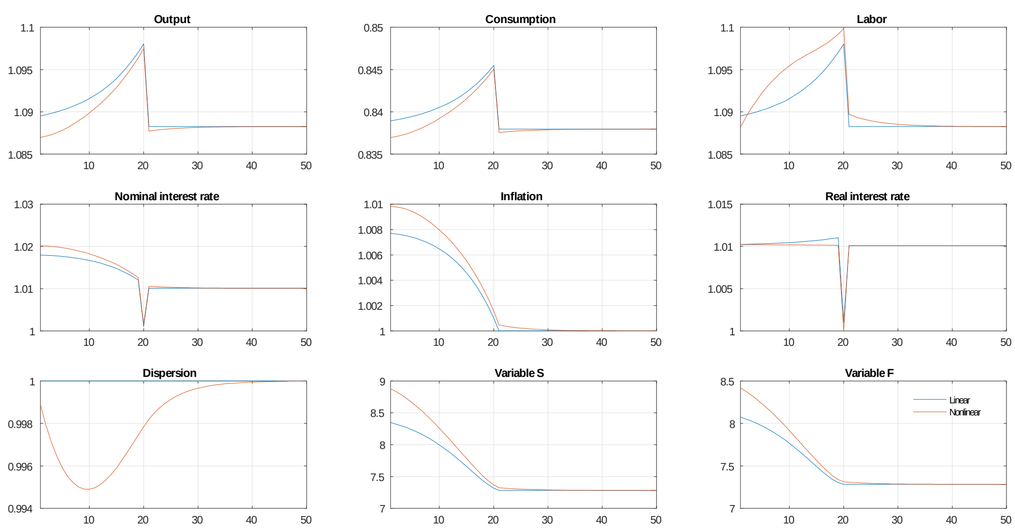

Figure D1. A comparison of linear and nonlinear solutions under $\phi_{E\pi} = 0$, $\phi_\pi = 1/\beta$ and $\phi_y = 0$.

We can see some qualitative differences between the linear and nonlinear solutions. For example, in the initial period, the nonlinear model predicts that output goes down, while the linear model predicts that it goes up. However, quantitatively, these differences are not very significant.

We explore a number of other parameterizations and obtain similar results. For example, for the Taylor rule with the output gap ($\phi_y = 0.5$) and persistence in the interest rate ($\mu = 0.82$), the difference between the linearized and nonlinear solutions is minimal, independently of whether the rule is parameterized by

[11] The linearized version of the model does not correspond exactly to the three-equation model studied before, e.g., the former includes government spendings and the endogenous natural level of output, the presence of which does not lead to the three-equation model; see Appendix A.

expected inflation or actual inflation . Moreover, similar to the linearized model (see Figure 3), we find that the nonlinear solutions are practically indistinguishable in two cases of $\phi_{E\pi} = 3$ and $\phi_{\pi} = 0$ and $\phi_{\pi} = 3$ and $\phi_{E\pi} = 0$.

7.0.2 Multiple sources of uncertainty

We next study how the introduction of more general sources of uncertainty affects the model's predictions about the effectiveness of FG. As described in Appendix A, we introduce six different shocks into the nonlinear model. As an example, in Figure D2, we introduce uncertainty in the nonlinear model which exhibited the FG puzzle, i.e., we parameterize the Taylor rule (52) by $\phi_{E\pi} \searrow 1$, $\phi_{\pi} = 0$; $\phi_y = 0$ and $\mu = 0$).

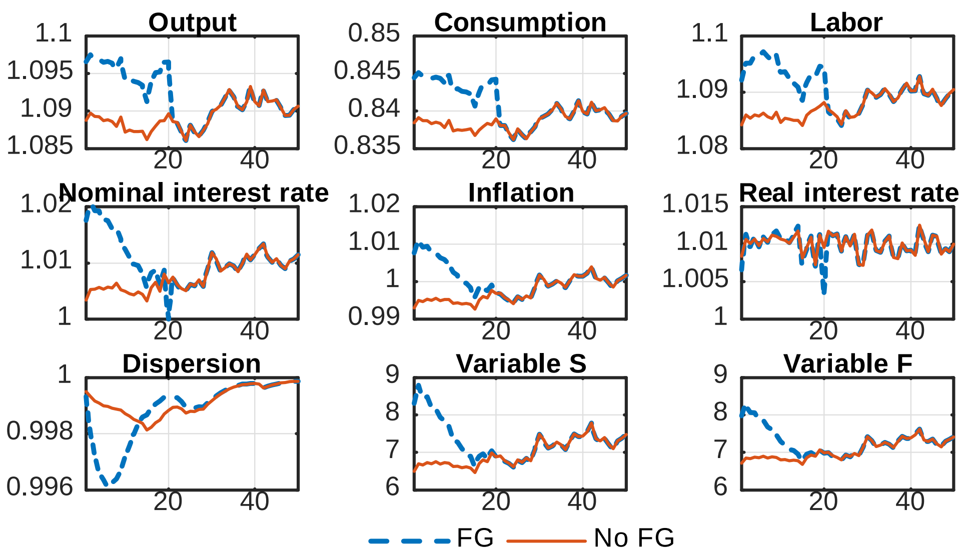

Figure D2. Forward guidance in the nonlinear stochastic model $\phi_{E\pi} \searrow 1$, $\phi_{\pi} = 0$ and $\phi_y = 0$.

The time series in Figure D2 look very similar to the FG puzzle dynamics in the corresponding deterministic model of Section 3.1. The output and inflation jump up in the initial period.

We also analyze the model with the Taylor rule that includes actual inflation $\phi_{E\pi} = 0$, $\phi_{\pi} \searrow 1$, $\phi_y = 0$ and $\mu = 0$ (not reported). The effect of the FG in the model with uncertainty is very similar to the one in the deterministic version of the model analyzed in Appendix C; see Figure C1. In our experiments, those parameterizations of the Taylor rule that led to backward stable solutions in the deterministic model also result in backward stable solutions in the model with uncertainty. Overall, we conclude that the introduction of uncertainty does not significantly affect the predictions of the model about the FG effectiveness.

Appendix E. New Keynesian model with capital

In this section, we extend the basic new Keynesian model described in Appendix A to include capital and provide additional sensitivity experiments.

E1. The model

We formulate the new Keynesian model with capital.

Households. A household j solves:

$$\max E_0 \sum_{t=0}^{\infty} \beta^t \exp\left(\eta_{u,t}\right) \left[\frac{C_t(j)^{1-\sigma} - 1}{1-\sigma} - \exp\left(\eta_{L,t}\right) \frac{L_t(j)^{1+\varphi} - 1}{1+\varphi} \right]$$

$$\text{s.t. } P_t C_t(j) + P_t \exp\left(\eta_{I,t}\right) I_t(j) + \frac{B_t(j)}{\exp\left(\eta_{B,t}\right) R_t} + T_t(j) = \tag{53}$$

$$B_{t-1}(j) + R_t^k K_{t-1}(j) + W_t L_t(j) + \Pi_t(j)$$

$$K_t(j) = (1-\delta) K_{t-1}(j) + I_t(j)$$

where $\beta \in (0,1)$ is the discount factor; σ and φ are the utility-function parameters; $\delta \in (0,1]$ is the depreciation rate of capital.

$\eta_{u,t}$ and $\eta_{L,t}$ are exogenous preference shocks: the former scales the overall momentary utility and the latter affects the marginal disutility of labor; $C_t(j)$, $L_t(j)$, $I_t(j)$, $K_{t-1}(j)$ and $B_t(j)$ are consumption, labor, investment, capital stock and nominal bonds holdings, respectively; P_t, W_t, R_t^k and R_t are the commodity price, nominal wage, (gross) nominal interest rate on capital and (gross) nominal interest rate on bonds, respectively; $\eta_{B,t}$ is an exogenous premium in the return to bonds (might reflect inefficiency in the financial sector, e.g., a risk premium required by households to hold a one-period bond); $\eta_{I,t}$ is an exogenous capital-embodied technology shock; $T_t(j)$ is lump-sum taxes; $\Pi_t(j)$ is the profit of intermediate-good producers. The exogenous shocks follow the following exogenous stochastic processes:

$$\eta_{u,t} = \rho^u \eta_{u,t-1} + \varepsilon_{u,t}, \qquad \varepsilon_{u,t} \sim \mathcal{N}\left(0, \sigma_u^2\right)$$

$$\eta_{L,t} = \rho^L \eta_{L,t-1} + \varepsilon_{L,t}, \qquad \varepsilon_{L,t} \sim \mathcal{N}\left(0, \sigma_L^2\right)$$

$$\eta_{B,t} = \rho^B \eta_{B,t-1} + \varepsilon_{B,t}, \qquad \varepsilon_{B,t} \sim \mathcal{N}\left(0, \sigma_B^2\right)$$

$$\eta_{I,t} = \rho^I \eta_{I,t-1} + \varepsilon_{I,t}, \qquad \varepsilon_{I,t} \sim \mathcal{N}\left(0, \sigma_I^2\right)$$

Final-good producers. They are the same as in the model without capital.

Intermediate-good producers. Technology of a producer i is

$$Y_t(i) = \exp\left(\eta_{a,t}\right) K_{t-1}(i)^{\alpha} L_t(i)^{1-\alpha}$$

where $L_t(i)$ is labor; $K_{t-1}(i)$ is capital; $\exp\left(\eta_{a,t}\right)$ is the productivity level that follows the exogenous stochastic process

$$\eta_{a,t} = \rho^a \eta_{a,t-1} + \varepsilon_{a,t}, \qquad \varepsilon_{a,t} \sim \mathcal{N}\left(0, \sigma_a^2\right)$$

Total cost (net of government subsidy v_t) in nominal terms:

$$\min_{K_{t-1}(i), L_t(i)} TC\left(Y_t(i)\right) = \left\{(1-v_t)\left(R_t^k K_{t-1}(i) + W_t L_t(i)\right)\right\} \tag{54}$$

$$\text{s.t. } Y_t(i) = \exp\left(\eta_{a,t}\right) K_{t-1}(i)^{\alpha} L_t(i)^{1-\alpha} \tag{55}$$

where W_t is the nominal wage rate.

A reoptimizing firm solves the same problem as in the model without capital.

Monetary authority. The monetary authority follows the same Taylor rule (40) as in the model without capital.

Natural level of output. The natural output level $Y_{N,t}$ can be determined from

$$\max E_0 \sum_{t=0}^{\infty} \beta^t \exp\left(\eta_{u,t}\right) \left[\frac{(C_t)^{1-\sigma}-1}{1-\sigma} - \exp\left(\eta_{L,t}\right) \frac{(L_t)^{1+\varphi}-1}{1+\varphi}\right]$$

$$\text{s.t. } C_t^* + \exp\left(\eta_{I,t}\right)\left[K_t - (1-\delta) K_{t-1}\right] = \exp\left(\eta_{a,t}\right) (K_{t-1})^{\alpha} (L_t)^{1-\alpha} - G_t \tag{56}$$

with $G_t = \left(1 - \frac{1}{\exp\left(\eta_{G,t}\right)}\right) Y_{N,t}$.

Summary of equilibrium conditions.

$$S_t = \frac{\exp\left(\eta_{u,t}+\eta_{L,t}\right)}{\exp\left(\eta_{a,t}\right)} C_t^{-\sigma} Y_t \frac{r_t^k}{\alpha} \left[\frac{K_{t-1}}{L_t}\right]^{1-\alpha} + \beta\theta E_t\left\{\pi_{t+1}^{\varepsilon} S_{t+1}\right\}, \tag{57}$$

$$F_t = \exp\left(\eta_{u,t}\right) C_t^{-\sigma} Y_t + \beta\theta E_t\left\{\pi_{t+1}^{\varepsilon-1} F_{t+1}\right\}, \tag{58}$$

$$\frac{S_t}{F_t} = \left[\frac{1-\theta\pi_t^{\varepsilon-1}}{1-\theta}\right]^{\frac{1}{1-\varepsilon}}, \tag{59}$$

$$\Delta_t = \left[(1-\theta)\left[\frac{1-\theta\pi_t^{\varepsilon-1}}{1-\theta}\right]^{\frac{\varepsilon}{\varepsilon-1}} + \theta\frac{\pi_t^{\varepsilon}}{\Delta_{t-1}}\right]^{-1}, \tag{60}$$

$$\exp\left(\eta_{u,t}\right) C_t^{-\sigma} = \beta \exp\left(\eta_{B,t}\right) R_t E_t\left\{\frac{\exp\left(\eta_{u,t+1}\right) C_{t+1}^{-\sigma}}{\pi_{t+1}}\right\}, \tag{61}$$

$$\exp\left(\eta_{u,t}+\eta_{I,t}\right) C_t^{-\sigma} = \beta E_t\left\{\exp\left(\eta_{u,t+1}\right) C_{t+1}^{-\sigma}\left[\exp\left(\eta_{I,t+1}\right)(1-\delta) + r_{t+1}^k\right]\right\}, \tag{62}$$

$$r_t^k = \frac{\alpha}{1-\alpha}\exp\left(\eta_{L,t}\right) L_t^{1+\varphi} C_t^{\sigma} K_{t-1}^{-1}, \tag{63}$$

$$Y_t = \exp\left(\eta_{a,t}\right) K_{t-1}^{\alpha} L_t^{1-\alpha} \Delta_t, \tag{64}$$

$$C_t + \exp\left(\eta_{I,t}\right)\left[K_t - (1-\delta) K_{t-1}\right] = \left(1 - \frac{\overline{G}}{\exp\left(\eta_{G,t}\right)}\right) Y_t, \tag{65}$$

where r_t^k is the marginal productivity of capital. There are 10 equations and 10 unknowns (C_t, L_t, K_t, Y_t, π_t, Δ_t, R_t, r_t^k, S_t, F_t). There are 7 exogenous shocks ($\eta_{a,t}$, $\eta_{u,t}$, $\eta_{L,t}$, $\eta_{B,t}$, $\eta_{R,t}$, $\eta_{G,t}$, $\eta_{I,t+1}$) and 3 endogenous state variables ($K_{t-1}, \Delta_{t-1}, R_{t-1}$).

E2. Numerical results

We finally study how the introduction of capital into the basic new Keynesian model affects the model's implications about the effectiveness of FG; see Appendix E for a description of such a model. In Figure E1, we show the non-linear solution under the Taylor rule (52) parameterized by either expected inflation

$\phi_{E\pi} = 2$, $\phi_\pi = 0$ or actual inflation $\phi_{E\pi} = 0$, $\phi_\pi = 2$ and the remaining coefficients are set at $\phi_y = 0.5$ and $\mu = 0.82$.

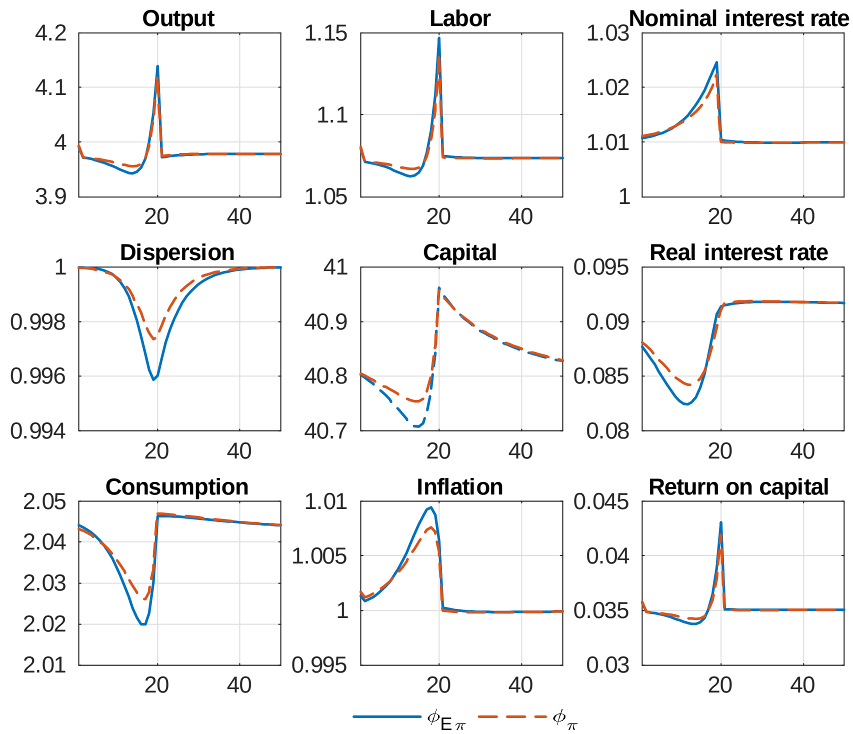

Figure E1. Model with capital: Taylor rules with expected inflation $\phi_{E\pi} = 2$, versus actual inflation $\phi_\pi = 2$ under $\phi_y = 0.5$ and $\mu = 0.82$.

With capital and the lagged nominal interest rate in the Taylor rule, the model has now two additional endogenous state variables. In this model, output responds not only to labor-input but also to capital-input changes. These differences do not affect qualitative implications of the model about the FG effectiveness: the series in Figure E.1 are backward stable and qualitatively similar to those in the baseline three equation model in Figure 4.

We perform a number of sensitivity experiments by varying the parameters in the Taylor rule (52). In our first sensitivity experiment, we assume the Taylor rule (40) has just actual inflation $\phi_\pi \searrow 1$. Recall that in the model without capital, this parameterization led to a version of the FG puzzle when just inflation reacts immediately, while the output gap increases gradually; see e.g., Figures D1 in Appendix D.

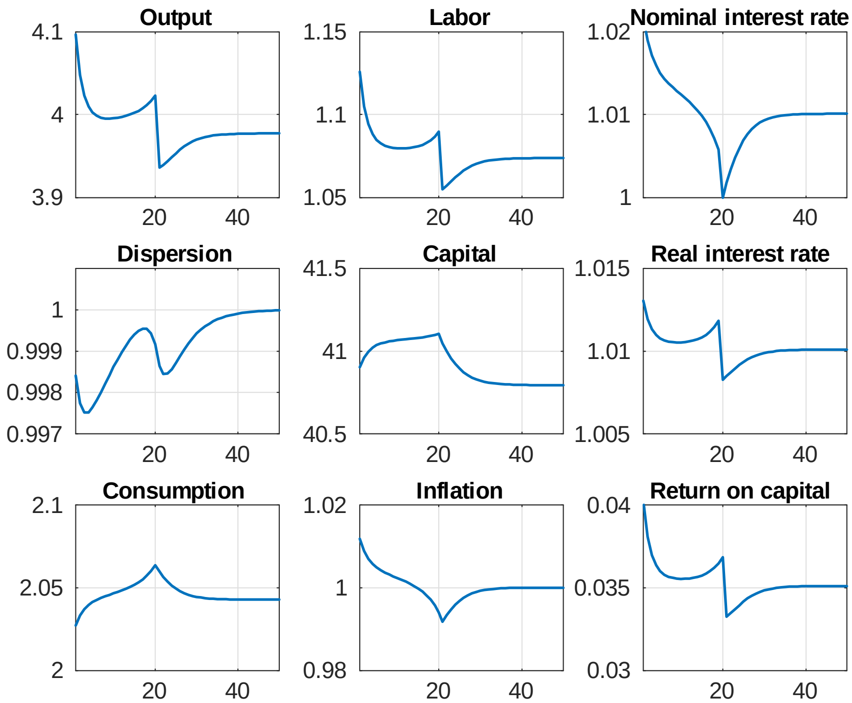

Figure E2. Model with capital: Taylor rule with actual inflation, $\phi_\pi = 1$, $\phi_{E\pi} = 0$, $\phi_y = 0$.

We observe some qualitative differences between the dynamics of the models with and without capital. As we can see from Figure E2, there are immediate effects of FG on both output and inflation, while consumption goes up slowly. Hence, we observe a stronger version of the FG puzzle than in the model without capital under this specific parameterization.

However, when ϕ_π in (40) increases, the FG puzzle disappears as it does in the model without capital.

In Figure E3, we show the model with capital in which $\phi_\pi = 3$.

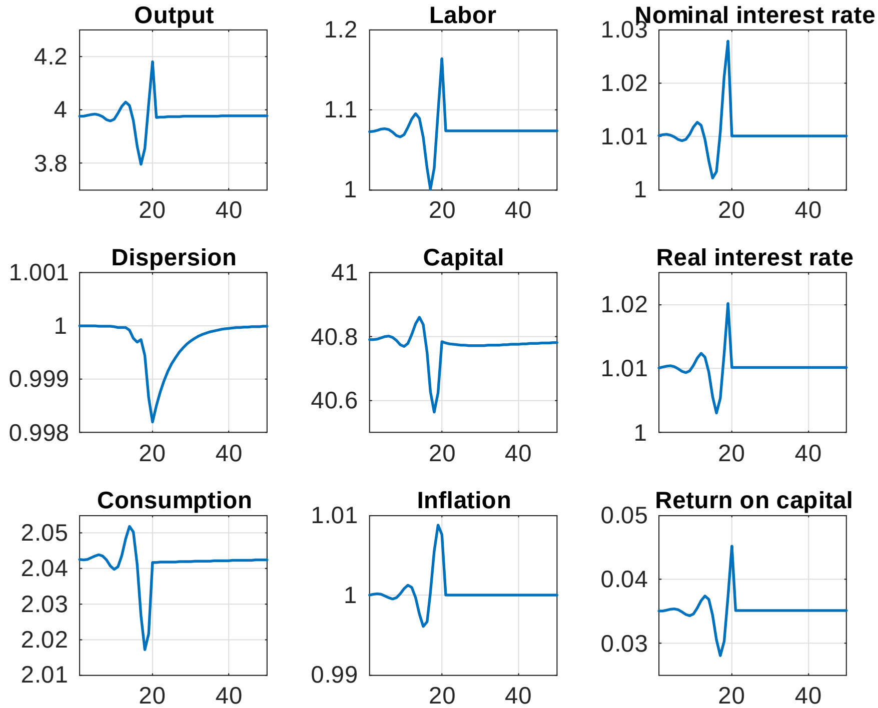

Figure E3. Model with capital: Taylor rule with actual inflation, $\phi_\pi = 3$, $\phi_{E\pi} = 0$, $\phi_y = 0$.

This case is qualitatively similar to the one reported for the baseline linear model in Figure D.2.

Finally, in Figure E4, we consider the FG puzzle scenario, i.e., we assume Taylor rule (40) with just expected inflation $\phi_{E\pi} \searrow 1$.

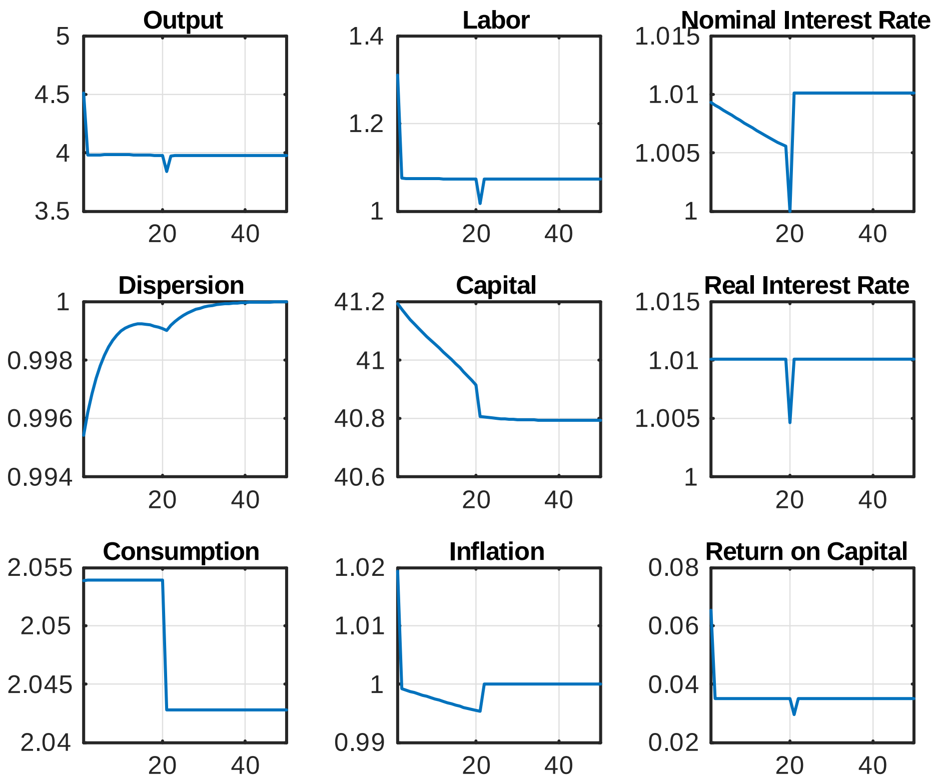

Figure E4. Model with capital: Taylor rule with expected inflation, $\phi_{E\pi} = 1$.

Unlike in the case of $\phi_\pi = 1$, FG is not very effective under $\phi_{E\pi} = 1$. As we can see from Figure E4, labor jumps up immediately in the first period and then it goes down in the second period, in spite of the smooth behavior of capital. The behavior of output here drastically differs from the one in the model without capital, namely, output jumps in the first period but goes back to the original level in the second period and remains there until the shock happens. Therefore, in terms of output, FG has no long-term effect but only a brief initial effect. It is surprising that consumption does not mimic output but behaves as in the basic model without capital, i.e., it jumps immediately and stays high until the shock happens. Hence, the consumption pattern looks resembles the FG puzzle in the baseline model. Thus, some version of the FG puzzle is observed in this case as well.

In sum, we do observe some qualitative differences between the model with capital and the basic linearized model in the main text (e.g., compare Figures E1-E4 for the model with capital with Figures 1-4 for the basic linear model). However, our most important finding remains unchanged: the Taylor rule with a weak response to inflation, e.g., $\phi_{E\pi} \searrow 1$ or $\phi_\pi \searrow 1$, produces backward explosive dynamics with some version of the FG puzzle, while a more responsive monetary policy (for example, with larger values of ϕ_π or with the inclusion of the output gap ϕ_y) eliminates the puzzle.