Supplementary Material

Supplement to "A tractable framework for analyzing a class of nonstationary Markov models"

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Appendix A: Turnpike theorem with Markov terminal condition

In this section, we introduce notation, provide several relevant definitions about random processes, and elaborate the proof of Theorem 2 (turnpike theorem) formulated in Section 2. The turnpike literature normally assumes a zero terminal capital for the finite-horizon economy, which is a convenient assumption for showing asymptotic convergence results. However, in applications, it is more effective to choose a terminal condition which is as close as possible to the infinite-horizon solution at T. This choice will make the finite-horizon approximation closer to the infinite-horizon solution. (In fact, if we guess the "exact" terminal condition on the infinite-horizon path, then the infinite- and finite-horizon trajectories would coincide.) Hence, we show our own version of the turnpike theorem for the growth model which holds for an arbitrary Markov terminal condition of the type $k_{T+1} = K_T(k_T, z_T)$, which extends the turnpike literature that focuses on a zero terminal condition $k_{T+1} = 0$.

Appendices A.1 and A.2 contain notations, definitions, and preliminaries. The proof of Theorem 2 relies on three lemmas presented in Appendices A.3–A.5. In Appendix A.3, we construct a limit program of a finite-horizon economy with a terminal condition $k_{T+1} = 0$; this construction is standard in the turnpike analysis (see Majumdar and Zilcha (1987), Mitra and Nyarko (1991), Joshi (1997)), and it is shown for the sake of completeness. In Appendix A.4, we prove a new result about convergence of the optimal program of the T-period stationary economy with an arbitrary terminal capital

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stock $k_{T+1} = K_T(k_T, z_T)$ to the limiting program of the finite-horizon economy with a zero terminal condition $k_{T+1} = 0$. In Appendix A.5, we show that the limit program of the finite-horizon economy with a zero terminal condition $k_{T+1} = 0$ is also an optimal program for the infinite-horizon nonstationary economy; in the proof, we also follow the previous turnpike literature. Finally, in Appendix A.6, we combine the results of Appendices A.3–A.5 to establish the claim of Theorem 2. Thus, our main theoretical contribution is contained in Appendix A.4.

A.1 *Notation and definitions*

Our exposition relies on standard measure theory notation; see, for example, Stokey, Lucas, and Prescott (1989), Santos (1999), and Stachurski (2009). Time is discrete and infinite, $t = 0, 1, \ldots$. Let $(\Omega, \mathcal{F}, P)$ be a probability space:

(a) $\Omega = \prod_{t=0}^{\infty} \Omega_t$ is a space of sequences $\epsilon \equiv (\epsilon_0, \epsilon_1 \ldots)$ such that $\epsilon_t \in \Omega_t$ for all t, where Ω_t is a compact metric space endowed with the Borel σ-field $\mathcal{E}_t$. Here, Ω_t is the set of all possible states of the environment at t and $\epsilon_t \in \Omega_t$ is the state of the environment at t.

(b) $\mathcal{F}$ is the σ-algebra on Ω generated by cylinder sets of the form $\prod_{\tau=0}^{\infty} A_\tau$, where $A_\tau \in \mathcal{E}_\tau$ for all τ and $A_\tau = \Omega_\tau$ for all but finitely many τ.

(c) P is the probability measure on $(\Omega, \mathcal{F})$.

We denote by $\{\mathcal{F}_t\}$ a filtration on Ω, where $\mathcal{F}_t$ is a sub σ-field of $\mathcal{F}$ induced by a partial history of environment $h_t = (\epsilon_0, \ldots, \epsilon_t) \in \prod_{\tau=0}^{t} \Omega_\tau$ up to period t, that is, $\mathcal{F}_t$ is generated by cylinder sets of the form $\prod_{\tau=0}^{t} A_\tau$, where $A_\tau \in \mathcal{E}_\tau$ for all $\tau \leq t$ and $A_\tau = \Omega_\tau$ for $\tau > t$. In particular, we have that $\mathcal{F}_0$ is the coarse σ-field $\{0, \Omega\}$, and that $\mathcal{F}_\infty = \mathcal{F}$. Furthermore, if Ω consists of either finite or countable states, ϵ is called a *discrete state process* or *chain*; otherwise, it is called a *continuous state process*. Our analysis focuses on continuous state processes; however, it can be generalized to chains with minor modifications.

We provide some definitions that will be useful for characterizing random processes; these definitions are standard and closely follow Stokey, Lucas, and Prescott (1989, Chapter 8.2).

Definition A1 (Stochastic process). A stochastic process on $(\Omega, \mathcal{F}, P)$ is an increasing sequence of σ-algebras $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \cdots \subseteq \mathcal{F}$; a measurable space $(Z, \mathcal{Z})$; and a sequence of functions $z_t : \Omega \to Z$ for $t \geq 0$ such that each z_t is $\mathcal{F}_t$ measurable.

Stationarity or time-homogeneity is an assumption that is commonly used in economic literature.

Definition A2 (Stationary process). A stochastic process z on $(\Omega, \mathcal{F}, P)$ is called stationary if the unconditional probability measure, given by

$$P_{t+1,\ldots,t+n}(C) = P\big(\{\epsilon \in \Omega : [z_{t+1}(\epsilon), \ldots, z_{t+n}(\epsilon)] \in C\}\big), \tag{S1}$$

is independent of t for all $C \in \mathcal{Z}^n$, $t \geq 0$, and $n \geq 1$.

A related notion is stationary (time-homogeneous) transition probabilities. Let us denote by $P_{t+1,\dots,t+n}(C|z_t = \bar{z}_t, \dots, z_0 = \bar{z}_0)$ the probability of the event $\{\epsilon \in \Omega : [z_{t+1}(\epsilon), \dots, z_{t+n}(\epsilon)] \in C\}$, given that the event $\{\epsilon \in \Omega : \bar{z}_t = z_t(\epsilon), \dots, \bar{z}_0 = z_0(\epsilon)\}$ occurs.

Definition A3 (Stationary transition probabilities). A stochastic process z on $(\Omega, \mathcal{F}, P)$ is said to have stationary transition probabilities if the conditional probabilities

$$P_{t+1,\dots,t+n}(C|z_t = \bar{z}_t, \dots, z_0 = \bar{z}_0) \tag{S2}$$

are independent of t for all $C \in \mathcal{Z}^n$, $\epsilon \in \Omega$, $t \geq 0$, and $n \geq 1$.

The assumption of stationary transition probabilities (S2) implies stationarity (S1) if the corresponding unconditional probability measures exist. However, a process can be nonstationary even if transition probabilities are stationary; for example, a unit root process or explosive process is nonstationary; see Stokey, Lucas, and Prescott (1989, Chapter 8.2) for a related discussion.

In general, $P_{t+1,\dots,t+n}(C)$ and $P_{t+1,\dots,t+n}(C|\cdot)$ depend on the entire history of the events up to t (i.e., the stochastic process z_t is measurable with respect to the sub σ-field $\mathcal{F}_t$). However, history-dependent processes are difficult to analyze. The literature distinguishes some special cases in which the dependence on history has relatively simple and tractable form. A well-known case is a class of Markov processes.

Definition A4 (Time-inhomogeneous Markov process). A stochastic process z on $(\Omega, \mathcal{F}, P)$ is (first-order) Markov if

$$P_{t+1,\dots,t+n}(C|z_t = \bar{z}_t, \dots, z_0 = \bar{z}_0) = P_{t+1,\dots,t+n}(C|z_t = \bar{z}_t), \tag{S3}$$

for all $C \in \mathcal{Z}^n$, $t \geq 0$, and $n \geq 1$.

The key property of a Markov process is that it is memoryless, namely, all past history $(z_t, \dots, z_0)$ is irrelevant for determining the future realizations except of the most recent past z_t. Note that the above definition does not require the Markov process to be time-homogeneous: it allows the functions $P_{t+1,\dots,t+n}(\cdot)$ to depend on time, as required by our analysis. Finally, if transition probabilities $P_{t+1,\dots,t+n}(C|z_t = \bar{z}_t)$ are independent of t for any $n \geq 1$, then the Markov process is time-homogeneous. If, in addition, there is an unconditional probability measure (S1), the resulting Markov process is stationary.

Definition A5 (Stationary Markov process). A stochastic process z on $(\Omega, \mathcal{F}, P)$ is called stationary Markov if the unconditional probability measure, given by

$$P_{t+1,\dots,t+n}(C) = P\big(\{\epsilon \in \Omega : z_{t+1}(\epsilon) \in C\}\big), \tag{S4}$$

is independent of t for all $C \in \mathcal{Z}^n$, $t \geq 0$, and $n \geq 1$.

Thus, time-homogeneous Markov process is stationary if it has time-homogeneous unconditional probability distribution.

A.2 *Infinite-horizon economy*

We consider an infinite-horizon nonstationary stochastic growth model in which preferences, technology, and laws of motion for exogenous variables change over time. The representative agent solves

$$\max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} E_0\left[\sum_{t=0}^{\infty} \beta^t u_t(c_t)\right] \tag{S5}$$

$$\text{s.t.} \quad c_t + k_{t+1} = (1-\delta)k_t + f_t(k_t, z_t), \tag{S6}$$

$$z_{t+1} = \varphi_t(z_t, \epsilon_{t+1}), \tag{S7}$$

where $c_t \geq 0$ and $k_t \geq 0$ denote consumption and capital, respectively; initial condition (k_0, z_0) is given; $u_t : \mathbb{R}_+ \to \mathbb{R}$ and $f_t : \mathbb{R}_+^2 \to \mathbb{R}_+$ and $\varphi_t : \mathbb{R}^2 \to \mathbb{R}$ are possibly time-dependent utility function, production functions, and law of motion for exogenous variable z_t, respectively; the sequence of u_t, f_t, and φ_t for $t \geq 0$ is known to the agent in period $t = 0$; ϵ_{t+1} is i.i.d; $\beta \in (0, 1)$ is the discount factor; $\delta \in [0, 1]$ is the depreciation rate; and $E_t[\cdot]$ is an operator of expectation, conditional on a t-period information set.

We make standard assumptions about the utility and production functions that ensure the existence, uniqueness, and interiority of a solution. Concerning the utility function u_t, we assume that for each $t \geq 0$, the following holds:

Assumption 1 (Utility function). (a) *u_t is twice continuously differentiable on $\mathbb{R}_+$*; (b) $u_t' > 0$, *that is, u_t is strictly increasing on $\mathbb{R}_+$, where* $u_t' \equiv \frac{\partial u_t}{\partial c}$; (c) $u_t'' < 0$, *that is, u_t is strictly concave on $\mathbb{R}_+$, where* $u_t'' \equiv \frac{\partial^2 u_t}{\partial c^2}$; *and* (d) *$u_t$ satisfies the Inada conditions* $\lim_{c\to 0} u_t'(c) = +\infty$ *and* $\lim_{c\to\infty} u_t'(c) = 0$.

Concerning the production function f_t, we assume that for each $t \geq 0$, the following holds:

Assumption 2 (Production function). (a) *f_t is twice continuously differentiable on $\mathbb{R}_+^2$*; (b) $f_t'(k, z) > 0$ *for all $k \in \mathbb{R}_+$ and $z \in \mathbb{R}_+$, where* $f_t' \equiv \frac{\partial f_t}{\partial k}$; (c) $f_t''(k, z) \leq 0$ *for all $k \in \mathbb{R}_+$ and $z \in \mathbb{R}_+$, where* $f_t'' \equiv \frac{\partial^2 f_t}{\partial k^2}$; *and* (d) *$f_t$ satisfies the Inada conditions* $\lim_{k\to 0} f_t'(k, z) = +\infty$ *and* $\lim_{k\to\infty} f_t'(k, z) = 0$ *for all $z \in \mathbb{R}_+$.*

We need one more assumption. Let us define a pure capital accumulation process $\{k_{t+1}^{\max}\}_{t=0}^{\infty}$ by assuming $c_t = 0$ for all t in (S6) which, for each history $h_t = (z_0, \ldots, z_t)$, leads to

$$k_{t+1}^{\max} = f_t\left(k_t^{\max}, z_t\right), \tag{S8}$$

where $k_0^{\max} \equiv k_0$. We impose an additional joint boundedness restriction on preferences and technology by using the constructed process (S8):

Assumption 3 (Objective function). $E_0[\sum_{t=0}^{\infty} \beta^t u_t(k_{t+1}^{\max})] < \infty$.

This assumption ensures that the objective function (S5) is bounded so that its maximum exists. In particular, Assumption 3 holds either (i) when u_t is bounded from above for all t, that is, $u_t(c) < \infty$ for any $c \geq 0$ or (ii) when f_t is bounded from above for all t, that is, $f_t(k, z_t) < \infty$ for any $k \geq 0$ and $z_t \in Z_t$. However, it also holds for economies with nonvanishing growth and unbounded utility and production functions as long as $u_t(k_{t+1}^{\max})$ does not grow too fast so that the product $\beta^t u_t(k_{t+1}^{\max})$ still declines at a sufficiently high rate and the objective function (S5) converges to a finite limit.

Definition A6 (Feasible program). A feasible program for the economy (S5)–(S7) is a pair of adapted (t-measurable) processes $\{c_t, k_{t+1}\}_{t=0}^{\infty}$ such that, given initial condition k_0, they satisfy $c_t \geq 0$, $k_{t+1} \geq 0$, and (S6) for each possible history $h_\infty = (\epsilon_0, \epsilon_1, \ldots)$.

We denote by $\Im^\infty$ a set of all feasible programs from given initial capital k_0. We next introduce the concept of solution to the model.

Definition A7 (Optimal program). A feasible program $\{c_t^\infty, k_{t+1}^\infty\}_{t=0}^{\infty} \in \Im^\infty$ is called optimal if

$$E_0\left[\sum_{t=0}^{\infty} \beta^t \left\{u_t\left(c_t^\infty\right) - u_t(c_t)\right\}\right] \geq 0 \tag{S9}$$

for every feasible process $\{c_t, k_{t+1}\}_{t=0}^{\infty} \in \Im^\infty$.

Stochastic models with time-dependent fundamentals are studied in Majumdar and Zilcha (1987), Mitra and Nyarko (1991), and Joshi (1997), among others. The existence results for this class of models have been established in the literature for a general measurable stochastic environment without imposing the restriction of Markov process (S7). In particular, this literature shows that, under Assumptions 1–3, there exists an optimal program $\{c_t^\infty, k_{t+1}^\infty\}_{t=0}^{\infty} \in \Im^\infty$ in the economy (S5)–(S7), and it is both interior and unique; see Theorem 4.1 in Mitra and Nyarko (1991) and see Theorem 7 in Majumdar and Zilcha (1987). The results of this literature apply to us as well.

A.3 *Limit program of finite-horizon economy with a zero terminal capital*

In this section, we consider a finite-horizon version of the economy (S5)–(S7) with a given terminal condition for capital $k_{T+1} = \kappa$. Specifically, we assume that the agent solves

$$\max_{\{c_t, k_{t+1}\}_{t=0}^{T}} E_0\left[\sum_{t=0}^{T} \beta^t u_t(c_t)\right] \tag{S10}$$

$$\text{s.t.} \quad \text{(S6), (S7),} \tag{S11}$$

where initial condition (k_0, z_0) and terminal condition $k_{T+1} = \kappa$ are given. We first define feasible programs for the finite-horizon economy.

Definition A8 (Feasible programs in the finite-horizon economy). A feasible program in the finite-horizon economy is a pair of adapted (i.e., $\mathcal{F}_t$-measurable for all t) processes $\{c_t, k_{t+1}\}_{t=0}^{T}$ such that, given initial condition k_0 and any partial history $h_T = (\epsilon_0, \ldots, \epsilon_T)$, they reach a given terminal condition $k_{T+1} = \kappa$ at T, satisfy $c_t \geq 0$, $k_{t+1} \geq 0$, and (S6), (S7) for all $t = 1, \ldots T$.

In this section, we focus on a finite-horizon economy that reaches a zero terminal condition, $k_{T+1} = 0$, at T. We denote by $\Im^{T,0}$ a set of all finite-horizon feasible programs from given initial capital k_0 and any partial history $h_T \equiv (\epsilon_0, \ldots, \epsilon_T)$ that attain given $k_{T+1} = 0$ at T. We next introduce the concept of solution for the finite-horizon model.

Definition A9 (Optimal program in the finite-horizon model). A feasible finite-horizon program $\{c_t^{T,0}, k_{t+1}^{T,0}\}_{t=0}^{T} \in \Im^{T,0}$ is called optimal if

$$E_0\left[\sum_{t=0}^{T} \beta^t \left\{u_t\left(c_t^{T,0}\right) - u_t(c_t)\right\}\right] \geq 0 \tag{S12}$$

for every feasible process $\{c_t, k_{t+1}\}_{t=0}^{T} \in \Im^{T,0}$.

The existence result for the finite-horizon version of the economy (S10), (S11) with a zero terminal condition is established in the literature. Namely, under Assumptions A1–A3, there exists an optimal program $\{c_t^{T,0}, k_{t+1}^{T,0}\}_{t=0}^{T} \in \Im^{T,0}$ and it is both interior and unique. The existence of the optimal program can be shown by using either a Bellman equation approach (see Mitra and Nyarko (1991, Theorem 3.1)) or a Euler equation approach (see Majumdar and Zilcha (1987, Theorems 1 and 2)).

We next show that under terminal condition $k_{T+1}^{T,0} = k_{T+1} = 0$, the optimal program in the finite-horizon economy (S10), (S11) has a well-defined limit.

Lemma 1. *A finite-horizon optimal program* $\{c_t^{T,0}, k_{t+1}^{T,0}\}_{t=0}^{T} \in \Im^{T,0}$ *with a zero terminal condition* $k_{T+1}^{T,0} = 0$ *converges to a limit program* $\{c_t^{\lim}, k_{t+1}^{\lim}\}_{t=0}^{\infty}$ *when* $T \to \infty$, *that is,*

$$k_{t+1}^{\lim} \equiv \lim_{T\to\infty} k_{t+1}^{T,0} \quad \text{and} \quad c_t^{\lim} \equiv \lim_{T\to\infty} c_t^{T,0}, \quad \text{for } t = 0, 1, \ldots. \tag{S13}$$

Proof. The existence of the limit program follows by three arguments (for any history):

(i) Extending time horizon from T to $T+1$ increases T-period capital of the finite-horizon optimal program, that is, $k_{T+1}^{T+1,0} > k_{T+1}^{T,0}$. To see this, note that the model with time horizon T has zero (terminal) capital $k_{T+1}^{T,0} = 0$ at T. When time horizon is extended from T to $T+1$, the model has zero (terminal) capital $k_{T+2}^{T+1,0} = 0$ at $T+1$ but it has strictly positive capital $k_{T+1}^{T+1,0} > 0$ at T; this follows by the Inada conditions—Assumption 1(d).

(ii) The optimal program for the finite-horizon economy has the following property of monotonicity with respect to the terminal condition: if $\{c'_t, k'_{t+1}\}_{t=0}^T$ and $\{c''_t, k''_{t+1}\}_{t=0}^T$ are two optimal programs for the finite-horizon economy with terminal conditions $\kappa' < \kappa''$, then the respective optimal capital choices have the same ranking in each period, that is, $k'_t \leq k''_t$ for all $t = 1, \dots, T$. This monotonicity result follows by either Bellman equation programming techniques (see Mitra and Nyarko (1991, Theorem 3.2 and Corollary 3.3)) or Euler equation programming techniques (see Majumdar and Zilcha (1987, Theorem 3)) or lattice programming techniques (see Hopenhayn and Prescott (1992)); see also Joshi (1997, Theorem 1) for generalizations of these results to non-convex economies. Hence, the stochastic process $\{k_{t+1}^{T,0}\}_{t=0}^T$ shifts up (weakly) in a pointwise manner when T increases to $T+1$, that is, $k_{t+1}^{T+1,0} \geq k_{t+1}^{T,0}$ for $t \geq 0$.

(iii) By construction, the capital program from the optimal program $\{c_{t+1}^{T,0}, k_{t+1}^{T,0}\}_{t=0}^T$ is bounded from above by the capital accumulation process $\{0, k_{t+1}^{\max}\}_{t=0}^T$ defined in (S8), that is, $k_{t+1}^{T,0} \leq k_{t+1}^{\max} < \infty$ for $t \geq 0$. A sequence that is bounded and monotone is known to have a well-defined limit. □

A.4 *Limit program of the T-period stationary economy*

We now show that the optimal program of the T-period stationary economy, introduced in Section 4, converges to the same limit program (S13) as the optimal program of the finite-horizon economy (S10), (S11) with a zero terminal condition. We denote by $\Im^{T,\kappa}$ a set of all feasible finite-horizon programs that attain a terminal condition $\kappa \neq 0$ of the T-period stationary economy. (We assume the same initial capital (k_0, z_0) and the same partial history $h_T \equiv (\epsilon_0, \dots, \epsilon_T)$ as those fixed for the finite-horizon economy (S10), (S11).)

Lemma 2. *The optimal program of the T-period stationary economy $\{c_t^{T,\kappa}, k_{t+1}^{T,\kappa}\}_{t=0}^T \in \Im^{T,\kappa}$ converges to a unique limit program $\{c_t^{\lim}, k_{t+1}^{\lim}\}_{t=0}^\infty \in \Im^\infty$ defined in* (S13) *as $T \to \infty$that is, for all $t \geq 0$,*

$$k_{t+1}^{\lim} \equiv \lim_{T\to\infty} k_{t+1}^{T,\kappa} \quad \textit{and} \quad c_t^{\lim} \equiv \lim_{T\to\infty} c_t^{T,\kappa}. \tag{S14}$$

Proof. The proof of the lemma follows by six arguments (for any history).

(i) Observe that, by Assumptions 1 and 2, the optimal program of the T-period stationary economy has a positive capital stock $k_{t+1}^{T,\kappa} > 0$ at T (since the terminal capital is generated by the capital decision function of a stationary version of the model), while for the optimal program $\{c_t^{T,0}, k_{t+1}^{T,0}\}_{t=0}^T \in \Im^{T,0}$ of the finite-horizon economy, it is zero by definition, $k_{T+1}^{T,0} = 0$.

(ii) The property of monotonicity with respect to terminal condition implies that if $k_{T+1}^{T,\kappa} > k_{T+1}^{T,0}$, then $k_{t+1}^{T,\kappa} \geq k_{t+1}^{T,0}$ for all $t = 1, \dots, T$; see our discussion in (ii) of the proof to Lemma 1.

(iii) Let us fix some $\tau \in \{1, \dots, T\}$. We show that up to period τ, the optimal program $\{c_t^{T,\kappa}, k_{t+1}^{T,\kappa}\}_{t=0}^{\tau}$ does not give higher expected utility than $\{c_t^{T,0}, k_{t+1}^{T,0}\}_{t=0}^{\tau}$, that is,

$$E_0\left[\sum_{t=0}^{\tau} \beta^t \left\{u_t\left(c_t^{T,\kappa}\right) - u_t\left(c_t^{T,0}\right)\right\}\right] \leq 0. \tag{S15}$$

Toward contradiction, assume that it does, that is,

$$E_0\left[\sum_{t=0}^{\tau} \beta^t \left\{u_t\left(c_t^{T,\kappa}\right) - u_t\left(c_t^{T,0}\right)\right\}\right] > 0. \tag{S16}$$

Then, consider a new process $\{c'_t, k'_{t+1}\}_{t=0}^{\tau}$ that follows $\{c_t^{T,\kappa}, k_{t+1}^{T,\kappa}\}_{t=0}^{T} \in \Im^{T,\kappa}$ up to period $\tau - 1$ and that drops down at τ to match $k_{\tau+1}^{T,0}$ of the finite-horizon program $\{c_t^{T,\kappa}, k_{t+1}^{T,\kappa}\}_{t=0}^{T} \in \Im^{T,0}$, that is, $\{c'_t, k'_{t+1}\}_{t=0}^{\tau} \equiv \{c_t^{T,\kappa}, k_{t+1}^{T,\kappa}\}_{t=0}^{\tau-1} \cup \{c_\tau^T + k_{\tau+1}^T - k_{\tau+1}^{T,0}, k_{\tau+1}^{T,0}\}$. By monotonicity (ii), we have $k_{\tau+1}^T - k_{\tau+1}^{T,0} \geq 0$, so that

$$\begin{aligned} &E_0\left[\sum_{t=0}^{\tau} \beta^t \left\{u_t\left(c'_t\right) - u_t\left(c_t^{T,\kappa}\right)\right\}\right] \\ &\quad = E_0\left[\beta^\tau \left\{u_t\left(c_\tau^T + k_{\tau+1}^T - k_{\tau+1}^{T,0}\right) - u_t\left(c_\tau^T\right)\right\}\right] \geq 0, \end{aligned} \tag{S17}$$

where the last inequality follows by Assumption 1(b) of strictly increasing u_t.

(iv) By construction, $\{c'_t, k'_{t+1}\}_{t=0}^{\tau}$ and $\{c_t^{T,0}, k_{t+1}^{T,0}\}_{t=0}^{\tau}$ reach the same capital $k_{\tau+1}^{T,0}$ at τ. Let us extend the program $\{c'_t, k'_{t+1}\}_{t=0}^{\tau}$ to T by assuming that it follows the process $\{c_t^{T,0}, k_{t+1}^{T,0}\}_{t=0}^{T}$ from the period $\tau + 1$ up to T, that is, $\{c'_t, k'_{t+1}\}_{t=\tau+1}^{T} \equiv \{c_t^{T,0}, k_{t+1}^{T,0}\}_{t=\tau+1}^{T}$. Then, we have

$$\begin{aligned} E_0\left[\sum_{t=0}^{T} \beta^t \left\{u_t\left(c'_t\right) - u_t\left(c_t^{T,0}\right)\right\}\right] &= E_0\left[\sum_{t=0}^{\tau} \beta^t \left\{u_t\left(c'_t\right) - u_t\left(c_t^{T,0}\right)\right\}\right] \\ &\geq E_0\left[\sum_{t=0}^{\tau} \beta^t \left\{u_t\left(c_t^{T,\kappa}\right) - u_t\left(c_t^{T,0}\right)\right\}\right] > 0, \end{aligned} \tag{S18}$$

where the last two inequalities follow by Result (S17) and Assumption (S16), respectively. Thus, we obtain a contradiction: The constructed program $\{c'_t, k'_{t+1}\}_{t=0}^{T} \in \Im^{T,0}$ is feasible in the finite-horizon economy with a zero terminal condition, $k'_{T+1} = 0$, and it gives strictly higher expected utility than the optimal program $\{c_t^{T,0}, k_{t+1}^{T,0}\}_{t=0}^{T} \in \Im^{T,0}$ in that economy.

(v) Holding τ fixed, we compute the limit of (S15) by letting T go to infinity:

$$\begin{aligned} &\lim_{T\to\infty} E_0\left[\sum_{t=0}^{\tau} \beta^t \left\{u_t\left(c_t^{T,\kappa}\right) - u_t\left(c_t^{T,0}\right)\right\}\right] \\ &\quad = \lim_{T\to\infty} E_0\left[\sum_{t=0}^{\tau} \beta^t u_t\left(c_t^{T,\kappa}\right)\right] - E_0\left[\sum_{t=0}^{\tau} \beta^t u_t\left(c_t^{\lim}\right)\right] \leq 0. \end{aligned} \tag{S19}$$

(vi) The last inequality implies that for any $\tau \geq 1$, the limit program $\{c_t^{\lim}, k_{t+1}^{\lim}\}_{t=0}^{\infty} \in \Im^{\infty}$ of the finite-horizon economy $\{c_t^{T,0}, k_{t+1}^{T,0}\}_{t=0}^{T} \in \Im^{T,0}$ with a zero terminal condition $k_T^{T,0} = 0$ gives at least as high expected utility as the optimal limit program $\{c_t^{T,\kappa}, k_{t+1}^{T,\kappa}\}_{t=0}^{T} \in \Im^{T,\kappa}$ of the T-period stationary economy. Since Assumptions 1 and 2 imply that the optimal program is unique, we conclude that $\{c_t^{\lim}, k_{t+1}^{\lim}\}_{t=0}^{\infty} \in \Im^{\infty}$ defined in (S13) is a unique limit of the optimal program $\{c_t^{T,\kappa}, k_{t+1}^{T,\kappa}\}_{t=0}^{T} \in \Im^{T,\kappa}$ of the T-period stationary economy. □

A.5 *Convergence of the finite-horizon economy to the infinite-horizon economy*

We now show a connection between the optimal programs of the finite-horizon and infinite-horizon economies. Namely, we show that the finite-horizon economy (S10), (S11) with a zero terminal condition $k_{T+1}^{T,0} = 0$ converges to the nonstationary infinite-horizon economy (S5)–(S7) as $T \to \infty$ provided that we fix the same initial condition k_0 and partial history $h_T = (\epsilon_0, \ldots, \epsilon_T)$ for both economies.

Lemma 3. *The limit program* $\{c_t^{\lim}, k_{t+1}^{\lim}\}_{t=0}^{\infty}$ *is a unique optimal program* $\{c_t^{\infty}, k_{t+1}^{\infty}\}_{t=0}^{\infty} \in \Im^{\infty}$ *in the infinite-horizon nonstationary economy* (S5)–(S7).

Proof. We prove this lemma by contradiction. We use the arguments that are similar to those used in the proof of Lemma 2.

(i) Toward contradiction, assume that $\{c_t^{\lim}, k_{t+1}^{\lim}\}_{t=0}^{\infty}$ is not an optimal program of the infinite-horizon economy $\{c_t^{\infty}, k_{t+1}^{\infty}\}_{t=0}^{\infty} \in \Im^{\infty}$. By definition of limit, there exists a real number $\varepsilon > 0$ and a subsequence of natural numbers $\{T_1, T_2, \ldots\} \subseteq \{0, 1, \ldots\}$ such that $\{c_t^{\infty}, k_{t+1}^{\infty}\}_{t=0}^{\infty} \in \Im^{\infty}$ gives strictly higher expected utility than the limit program of the finite-horizon economy $\{c_t^{\lim}, k_{t+1}^{\lim}\}_{t=0}^{\infty}$, that is,

$$E_0\left[\sum_{t=0}^{T_n} \beta^t \left\{u_t\left(c_t^{\infty}\right) - u_t\left(c_t^{\lim}\right)\right\}\right] > \varepsilon \quad \text{for all } T_n \in \{T_1, T_2, \ldots\}. \tag{S20}$$

(ii) Let us fix some $T_n \in \{T_1, T_2, \ldots\}$ and consider any finite $T \geq T_n$. Assumptions 1 and 2 imply that $k_{T+1}^{\infty} > 0$, while $k_{T+1}^{T,0} = 0$ by definition of the finite-horizon economy with a zero terminal condition. The monotonicity of the optimal program with respect to a terminal condition implies that if $k_{T+1}^{\infty} > k_{T+1}^{T,0}$, then $k_{t+1}^{\infty} \geq k_{t+1}^{T,0}$ for all $t = 1, \ldots, T$; see our discussion in (ii) of the proof of Lemma 1.

(iii) Following the arguments in (iii) of the proof of Lemma 2, we can show that up to period T_n, the optimal program $\{c_t^{\infty}, k_{t+1}^{\infty}\}_{t=0}^{T_n}$ does not give higher expected utility than $\{c_t^{T,0}, k_{t+1}^{T,0}\}_{t=0}^{T_n}$, that is,

$$E_0\left[\sum_{t=0}^{T_n} \beta^t \left\{u_t\left(c_t^{\infty}\right) - u_t\left(c_t^{T,0}\right)\right\}\right] \leq 0 \quad \text{for all } T_n. \tag{S21}$$

(iv) Holding T_n fixed, we compute the limit of (S21) by letting T go to infinity:

$$\lim_{T\to\infty} E_0\left[\sum_{t=0}^{T_n} \beta^t\left\{u_t\left(c_t^{\infty}\right) - u_t\left(c_t^{T,0}\right)\right\}\right]$$

$$= E_0\left[\sum_{t=0}^{T_n} \beta^t u_t\left(c_t^{\infty}\right) - \beta^t u_t\left(c_t^{\lim}\right)\right] \le 0 \quad \text{for all } T_n. \tag{S22}$$

However, result (S22) contradicts our assumption in (S20).

(v) We conclude that for any subsequence $\{T_1, T_2, \dots\} \subseteq \{0, 1, \dots\}$, we have

$$E_0\left[\sum_{t=0}^{T_n} \beta^t\left\{u_t\left(c_t^{\infty}\right) - u_t\left(c_t^{\lim}\right)\right\}\right] \le 0 \quad \text{for all } T_n. \tag{S23}$$

However, under Assumptions 1 and 2, the optimal program $\{c_t^{\infty}, k_{t+1}^{\infty}\}_{t=0}^{\infty} \in \Im^{\infty}$ is unique, and hence, it must be that $\{c_t^{\infty}, k_{t+1}^{\infty}\}_{t=0}^{\infty}$ coincides with $\{c_t^{\lim}, k_{t+1}^{\lim}\}_{t=0}^{\infty}$ for all $t \ge 0$. □

A.6 *Proof of the turnpike theorem*

We now combine the results of Lemmas 1–3 together into a turnpike-style theorem to show the convergence of the optimal program of the T-period stationary economy to that of the infinite-horizon nonstationary economy. To be specific, Lemma 1 shows that the optimal program of the finite-horizon economy with a zero terminal condition $\{c_t^{T,0}, k_{t+1}^{T,0}\}_{t=0}^{T} \in \Im^{T,0}$ converges to the limit program $\{c_t^{\lim}, k_{t+1}^{\lim}\}_{t=0}^{\infty}$. Lemma 2 shows that the optimal program of the T-period stationary economy $\{c_t^{T,\kappa}, k_{t+1}^{T,\kappa}\}_{t=0}^{T}$ also converges to the same limit program $\{c_t^{\lim}, k_{t+1}^{\lim}\}_{t=0}^{\infty}$. Finally, Lemma 3 shows that the limit program of the finite-horizon economies $\{c_t^{\lim}, k_{t+1}^{\lim}\}_{t=0}^{\infty}$ is optimal in the nonstationary infinite-horizon economy. Then, it must be the case that the limit optimal program of the T-period stationary economy $\{c_t^{T,\kappa}, k_{t+1}^{T,\kappa}\}_{t=0}^{T}$ is optimal in the infinite-horizon nonstationary economy. This argument is formalized below.

Proof of Theorem 2 (Turnpike theorem). The proof follows by definition of limit and Lemmas 1–3. Let us fix a real number $\varepsilon > 0$ and a natural number τ such that $1 \le \tau < \infty$ and consider a possible partial history $h_T = (\epsilon_0, \dots, \epsilon_T)$.

(i) Lemma 1 shows that $\{c_t^{T,0}, k_{t+1}^{T,0}\}_{t=0}^{T} \in \Im^{T,0}$ converges to a limit program $\{c_t^{\lim}, k_{t+1}^{\lim}\}_{t=0}^{\infty}$ as $T \to \infty$. Then, definition of limit implies that there exists $T_1(h_T) > 0$ such that $|k_{t+1}^{T,0} - k_{t+1}^{\lim}| < \frac{\varepsilon}{3}$ for $t = 0, \dots, \tau$.

(ii) Lemma 2 implies that the finite-horizon problem of the T-period stationary economy $\{c_t^{T,\kappa}, k_{t+1}^{T,\kappa}\}_{t=0}^{T}$ also converges to limit program $\{c_t^{\lim}, k_{t+1}^{\lim}\}_{t=0}^{\infty}$ as $T \to \infty$. Then, there exists $T_2(h_T) > 0$ such that $|k_{t+1}^{\lim} - k_{t+1}^{T,\kappa}| < \frac{\varepsilon}{3}$ for $t = 0, \dots, \tau$.

(iii) Lemma 3 implies the program $\{c_t^{T,0}, k_{t+1}^{T,0}\}_{t=0}^{T} \in \Im^{T,0}$ converges to the infinite-horizon optimal program $\{c_t^{\infty}, k_{t+1}^{\infty}\}_{t=0}^{\infty}$ as $T \to \infty$. Then, there exists $T_3(h_T) > 0$ such that $|k_{t+1}^{T,0} - k_{t+1}^{\infty}| < \frac{\varepsilon}{3}$ for $t = 0, \dots, \tau$.

(iv) Then, the triangular inequality implies

$$\begin{aligned}\left|k_{t+1}^{T,\kappa}-k_{t+1}^{\infty}\right| &= \left|k_{t+1}^{T,\kappa}-k_{t+1}^{\lim}+k_{t+1}^{\lim}-k_{t+1}^{T,0}+k_{t+1}^{T,0}-k_{t+1}^{\infty}\right| \\ &\leq \left|k_{t+1}^{T,\kappa}-k_{t+1}^{\lim}\right|+\left|k_{t+1}^{\lim}-k_{t+1}^{T,0}\right|+\left|k_{t+1}^{T,0}-k_{t+1}^{\infty}\right| < \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon,\end{aligned}$$

for $T(h_T) \geq \max\{T_1(h_T), T_2(h_T), T_3(h_T)\}$.

(v) Finally, consider all possible partial histories $\{h_T\}$ and define $T^*(\varepsilon, \tau, x_T^T) \equiv \max_{\{h_T\}} T(h_T)$. By construction, for any $T > T^*(\varepsilon, \tau, x_T^T)$, the result of the theorem holds. □

Remark A1. Our proof of the turnpike theorem addresses a technical issue that does not arise in the literature that focuses on finite-horizon economies with a zero terminal condition; see, for example, Majumdar and Zilcha (1987), Mitra and Nyarko (1991), and Joshi (1997). Their construction relies on the fact that the optimal program of the finite-horizon economy is always pointwise below the optimal program of the infinite-horizon economy, that is, $k_{t+1}^{T,\kappa} \leq k_{t+1}^{\infty}$, for $t = 1, \ldots, \tau$, and it gives strictly higher expected utility up to T than does the infinite-horizon optimal program (because excess capital can be consumed at terminal period T). This argument does not directly apply to our T-period stationary economy: our finite-horizon program can be either below or above the infinite-horizon program depending on a specific T-period terminal condition. Our proof addresses this issue by constructing in Lemma 2 a separate limit program for the T-period stationary economy.

Appendix B: Implementation of EFP for growth model

In this section, we describe the implementation of the EFP method used to produce the numerical results in the main text.

Algorithm 1a ((Implementation): Extended function path (EFP) for the growth model).

The goal of EFP.

EFP is aimed at approximating a solution of a nonstationary model during the first τ periods, that is, it finds approximating functions $(\widehat{K}_0, \ldots, \widehat{K}_\tau)$ such that $\widehat{K}_t \approx K_t$ for $t = 1, \ldots \tau$, where K_t and $\widehat{K}_t$ are a t-period true capital function and its parametric approximation, respectively.

Step 0. Initialization.

a. Choose time horizon $T \gg \tau$ for constructing T-period stationary economy.

b. Construct a deterministic path $\{z_t^*\}_{t=0}^T$ for exogenous state variable $\{z_t\}_{t=0}^T$ satisfying $z_{t+1}^* = \varphi_t(z_t^*, E_t[\epsilon_{t+1}])$ for $t = 0, \ldots, T$.

c. Construct a deterministic path $\{k_t^*\}_{t=0}^T$ for endogenous state variable $\{k_t\}_{t=0}^T$ satisfying

$u_t'(c_t^*) = \beta u_t'(c_{t+1}^*)(1 - \delta + f_{t+1}'(k_{t+1}^*, z_{t+1}^*))$.

$c_t^* + k_{t+1}^* = (1 - \delta)k_t^* + f_t(k_t^*, z_t^*)$ for $t = 0, \ldots, T$.

d. For $t = 0, \dots, T$:
 Construct a grid $\{(k_{m,t}, z_{m,t})\}_{m=1}^{M}$ centered at (k_t^*, z_t^*).
 Choose integration nodes, $\epsilon_{j,t}$, and weights, $\omega_{j,t}$, for $j = 1, \dots, J$.
 Construct future shocks $z'_{m,j,t} = \varphi_t(z_{m,t}, \epsilon_{j,t})$.

e. Write a t-period discretized system of the optimality conditions:

(i) $u'_t(c_{m,t}) = \beta \sum_{j=1}^{J} \omega_{j,t}[u'_t(c'_{m,j,t})\{1 - \delta + f_{t+1}(k'_{m,t}, z'_{m,j,t})\}]$.

(ii) $c_{m,t} + k'_{m,t} = (1 - \delta)k_{m,t} + f_t(k_{m,t}, z_{m,t})$.

(iii) $c'_{m,j,t} + k''_{m,j,t} = (1 - \delta)k'_{m,t} + f_{t+1}(k'_{m,t}, z'_{m,j,t})$.

(iv) $k'_{m,t} = \widehat{K}_t(k_{m,t}, z_{m,t})$ and $k''_{m,j,t} = \widehat{K}_{t+1}(k'_{m,t}, z'_{m,j,t})$.

d. Assume that the model becomes stationary at T.

Step 1: Terminal condition.
Find $\widehat{K}_T = \widehat{K}_{T+1}$ that approximately solves the system (i)–(iv) on the grid for the T-period stationary economy $f_{T+1} = f_T$, $u_{T+1} = u_T$, $\varphi_{T+1} = \varphi_T$.

Step 2: Backward induction.
Construct the function path $(\widehat{K}_0, \dots, \widehat{K}_{T-1}, \widehat{K}_T)$ that approximately solves the system (i)–(iv) for each $t = 0, \dots, T$ and that matches the given terminal function $\widehat{K}_T$ constructed in Step 1.

Step 3: Turnpike property.
Simulate the process $\widehat{K}_0$ and use a subset of simulated points as initial conditions (k_0, z_0). For each initial condition, draw a history $h_\tau = (\epsilon_0, \dots, \epsilon_\tau)$. Use the decision functions $(\widehat{K}_0, \dots, \widehat{K}_\tau)$ to simulate the economy's trajectories $(k_0^T, \dots k_\tau^T)$. Check that the trajectories converge to a unique limit $\lim_{T\to\infty}(k_0^T, \dots k_\tau^T) = (k_0^*, \dots k_\tau^*)$ by constructing $(K_0, \dots, K_T)$ under different T and K_T.

The EFP solution:
Use $(\widehat{K}_0, \dots, \widehat{K}_\tau)$ as an approximation to $(K_0, \dots, K_\tau)$ and discard the remaining $T - \tau$ functions.

The EFP method is more expensive than conventional solution methods for stationary models because decision functions must be constructed not just once but for T periods. We implement EFP in the way that keeps its cost relatively low: First, to approximate decision functions, we use a version of the Smolyak (sparse) grid technique. Specifically, we use a version of the Smolyak method that combines a Smolyak grid with ordinary polynomials for approximating functions off the grid. This method was described in Judd, Maliar, Maliar, and Valero (2014) who found it to be sufficiently accurate in the context of a similar growth model, namely, unit-free residuals in the model's equations do not exceed 0.01% on a stochastic simulation of 10,000 observations. For this version of the Smolyak method, the polynomial coefficients are overdetermined; for example, in a two-dimensional case, we have 13 points in a second-level Smolyak grid, and we have only six coefficients in second-degree ordinary polynomial. Hence, we identify the coefficients using a least-squares regression; we use an SVD decomposition, to enhance numerical stability; see Judd et al. (2014) for a discussion of this and other numerically stable approximation methods. We do not construct the Smolyak grid within a hypercube normalized to $[-1, 1]^2$, as do Smolyak methods that rely on Chebyshev polynomials used in, for example, Krueger and Kubler (2004), Malin, Krueger, and Kubler (2011), and Judd et al. (2014). Instead, we construct a sequence of Smolyak grids around actual steady state and thus, the hypercube, in which the Smolyak grid is constructed, grows over time as shown in Figures 1 and 8.

Second, to approximate expectation functions, we use Gauss–Hermite quadrature rule with 10 integration nodes. However, a comparison analysis in Judd et al. (2014) shows that for models with smooth decision functions like ours, the number of integration nodes plays only a minor role in the properties of the solution; for example, the results will be the same up to six digits of precision if, instead of ten integration nodes, we use just two nodes or a simple linear monomial rule (a two-node Gauss–Hermite quadrature rule is equivalent to a linear monomial integration rule for the two-dimensional case). However, simulation-based Monte Carlo-style integration methods produce very inaccurate approximations for integrals and are not considered in this paper; see Judd et al. (2014) for discussion.

Third, to solve for the coefficients of decision functions, we use a simple derivative-free fixed-point iteration method in line with Gauss–Jacobi iteration. Let us rewrite the Euler equation (i) constructed in the initialization step of the algorithm by premultiplying both sides by t-period capital:

$$\widehat{k}'_{m,t} = \beta \sum_{j=1}^{J} \epsilon_{j,t} \left[\frac{u'_t(c'_{m,j,t})}{u'_t(c_{m,t})} \left\{ 1 - \delta + f_{t+1}\left(k'_{m,t} k^*_{t+1}, z'_{m,j,t} z^*_{t+1}\right) \right\} \right] k'_{m,t}. \tag{S24}$$

We use different notation, $k'_{m,t}$ and $\widehat{k}'_{m,t}$, for t-period capital in the left- and right-hand side of (S24), respectively, in order to describe our fixed-point iteration method. Namely, we substitute $k'_{m,t}$ in the right-hand side of (S24) and in the constraints (ii) and (iii) in the initialization step to compute $c_{m,t}$ and $c'_{m,j,t}$, respectively, and we obtain a new set of values of the capital function on the grid $\widehat{k}'_{m,t}$ in the left-hand side. We iterate on these steps until convergence.

Our approximation functions $\widehat{K}_t$ are ordinary polynomial functions characterized by a time-dependent vector of parameters b_t, that is, $\widehat{K}_t = \widehat{K}(\cdot; b_t)$. So, operationally, the iteration is performed not on the grid values $k'_{m,t}$ and $\widehat{k}'_{m,t}$ but on the coefficients of the approximation functions. The iteration procedure differs in Steps 1 and 2.

In Step 1, we construct a solution to T-period stationary economy. For iteration i, we fix some initial vector of coefficients b, compute $k'_{m,T+1} = \widehat{K}(k_{m,T}, z_{m,T}; b)$, find $c_{m,T}$ and $c'_{m,j,T}$ to satisfy constraints (ii) and (iii), respectively, and find $\widehat{k}'_{m,T+1}$ from the Euler equation (i). We run a regression of $\widehat{k}'_{m,T+1}$ on $\widehat{K}(k_{m,T}, z_{m,T}; \cdot)$ in order to re-estimate the coefficients $\widehat{b}$ and we compute the coefficients for iteration $i+1$ as a weighted average, that is, $b^{(i+1)} = (1-\xi)b^{(i)} + \xi\widehat{b}^{(i)}$, where $\xi \in (0, 1)$ is a damping parameter (typically,

$\xi = 0.05$). We use partial updating instead of full updating $\xi = 1$ because fixed-point iteration can be numerically unstable and using partial updating enhances numerical stability; see Maliar et al. (2011). These kinds of fixed-point iterations are used by numerical methods that solve for equilibrium in conventional stationary Markov economies; see for example, Judd, Maliar, and Maliar (2011) and Judd et al. (2014).

In Step 2, we iterate on the path for the polynomial coefficients using Gauss–Jacobi-style iterations in line with Fair and Taylor (1983). Specifically, on iteration j, we take a path for the coefficients vectors $\{b_1^{(j)}, \ldots, b_T^{(j)}\}$, compute the corresponding path for capital quantities using $k'_{m,t} = \widehat{K}_t(k_{m,t}, z_{m,t}; b_t^{(j)})$, and find a path for consumption quantities $c_{m,t}$ and $c'_{m,j,t}$ from constraints (ii) and (iii), respectively, for $t = 0, \ldots, T$. Substitute these quantities in the right-hand side of a sequence of Euler equations for $t = 0, \ldots, T$ to obtain a new path for capital quantities in the left-hand side of the Euler equation $\widehat{k}'_{m,t}$ for $t = 0, \ldots, T-1$. Run $T-1$ regressions of $\widehat{k}'_{m,t}$ on polynomial functional forms $\widehat{K}_t(k_{m,t}, z_{m,t}; b_t)$ for $t = 0, \ldots, T-1$ to construct a new path for the coefficients $\{\widehat{b}_0^{(j)}, \ldots, \widehat{b}_{T-1}^{(j)}\}$. Compute the path of the coefficients for iteration $j+1$ as a weighted average, that is, $b_t^{(j+1)} = (1-\xi)b_t^{(j)} + \xi\widehat{b}_t^{(j)}$, $t = 0, \ldots, T-1$, where $\xi \in (0, 1)$ is a damping parameter which we again typically set at $\xi = 0.05$. (Observe that this iteration procedure changes all the coefficients on the path except of the last one $b_T^{(j)} \equiv b$, which is a given terminal condition that we computed in Step 1 from the T-period stationary economy.)

In fact, the problem of constructing a path for function coefficients is similar to the problem of constructing a path for variables: in both cases, we need to solve a large system of nonlinear equations. The difference is that under EFP, the arguments of this system are not variables but parameters of the approximating functions. Instead of Gauss–Jacobi-style iteration on path, we can use Gauss–Siedel fixed-point iteration (shooting), Newton-style solvers, or any other technique that can solve a system of nonlinear equations; see Lipton, Poterba, Sachs, and Summers (1980), Atolia and Buffie (2009a, 2009b), Heer and Maußner (2010), and Grüne, Semmler, and Stieler (2015) for examples of such techniques.

Let us now finally provide an additional illustration to the solution shown in Section 3.4. Specifically, in Figure 2, we plot a two-dimensional sequence of capital decision functions under fixed productivity level $z = 1$, while here we provide a three-dimensional plot of the same decision function for adding the productivity level. We again illustrate the capital functions for periods 1, 20, and 40 (i.e., $k_2 = K_1(k_1, z_1)$, $k_{21} = K_{20}(k_{20}, z_{20})$, and $k_{41} = K_{40}(k_{40}, z_{40})$) which we approximate using Smolyak (sparse) grids. In Step 1 of the algorithm, we construct the capital function K_{40} by assuming that the economy becomes stationary in period $T = 40$; and in Step 2, we construct a path of the capital functions that $(K_1, \ldots, K_{39})$ that matches the corresponding terminal function K_{40}. The Smolyak grids are shown by stars in the horizontal $k_t \times z_t$ plane. The domain for capital (on which Smolyak grids are constructed) and the range of the constructed capital function grow at the rate of labor-augmenting technological progress.

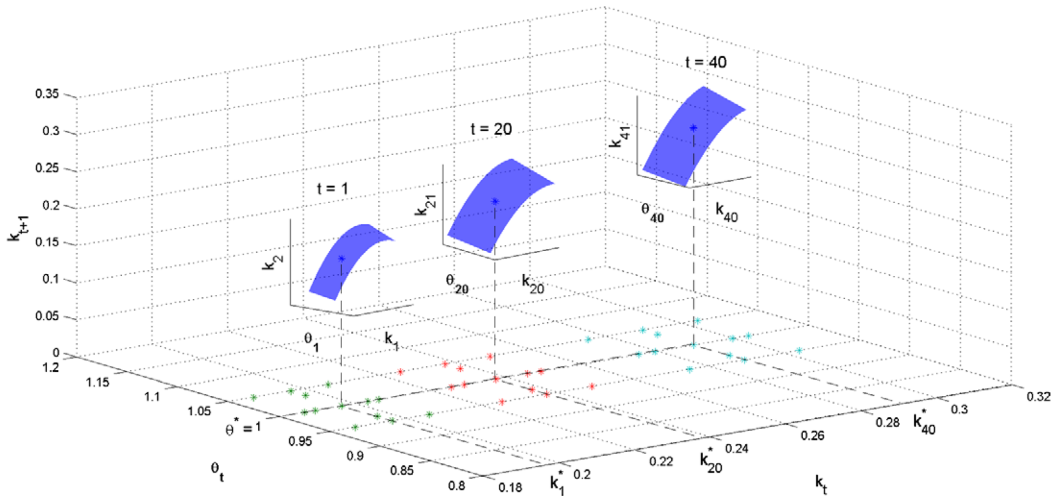

FIGURE S1. Function path, produced by EFP, for a growth model with technological progress.

APPENDIX C: PATH-SOLVING METHODS FOR NONSTATIONARY MODELS

We first describe the shooting method of Lipton et al. (1980) for a nonstationary deterministic economy, and we then elaborate the extended path (EP) of Fair and Taylor (1983) for a nonstationary economy with uncertainty.

Shooting methods To illustrate the class of shooting methods, let us substitute c_t and c_{t+1} from (S6) into the Euler equation of (S5)–(S7) to obtain a second-order difference equation,

$$u_t'\big((1-\delta)k_t + f_t(k_t, z_t) - k_{t+1}\big)$$
$$= \beta E_t\big[u_{t+1}'\big((1-\delta)k_{t+1} + f_{t+1}(k_{t+1}, z_{t+1}) - k_{t+2}\big)\big(1-\delta + f_{t+1}'(k_{t+1}, z_{t+1})\big)\big]. \quad \text{(S25)}$$

Initial condition (k_0, z_0) is given. Let us abstract from uncertainty by assuming that $z_t = 1$ for all t, choose a sufficiently large T, and fix some terminal condition k_{T+1} (typically, the literature assumes that the economy arrives in the steady state $k_{T+1} = k^*$).[1] To approximate the optimal path, we must solve numerically a system of T nonlinear equations (S25) with respect to T unknowns $\{k_1, \dots, k_T\}$. It is possible to solve the system (S25) by using a Newton-style or any other numerical solver. However, a more efficient alternative could be numerical methods that exploit the recursive structure of the system (S25) such as shooting methods (Gauss–Siedel iteration). There are two types of shooting methods: a forward shooting and a backward shooting. A typical forward shooting method expresses k_{t+2} in terms of k_t and k_{t+1} using (S25) and constructs a forward path $(k_1, \dots, k_{T+1})$; it iterates on k_1 until the path reaches a given terminal condition $k_{T+1} = k^*$. In turn, a typical reverse shooting method expresses k_t in terms of k_{t+1} and k_{t+2} and constructs a backward path $\{k_T, \dots, k_0\}$; it iterates on k_T until the

[1] The turnpike theorem implies that in initial τ periods, the optimal path is insensitive to a specific terminal condition used if $\tau \ll T$.

path reaches a given initial condition k_0. A shortcoming of shooting methods is that they tend to produce explosive paths, in particular, forward shooting methods; see Atolia and Buffie (2009a, 2009b) for a careful discussion and possible treatments of this problem.

Fair and Taylor (1983) method The EP method of Fair and Taylor (1983) allows us to solve nonstationary economic models with uncertainty by approximating expectation functions under the assumption of certainty equivalence. To see how this method works, consider the system (S25) with uncertainty and as an example, assume that z_{t+1} follows a possibly nonstationary Markov process $\ln(z_{t+1}) = \rho_t \ln(z_t) + \sigma_t \epsilon_{t+1}$, where the sequences $(\rho_0, \rho_1, \ldots)$ and $(\sigma_0, \sigma_1, \ldots)$ are deterministically given at $t = 0$ and $\epsilon_{t+1} \sim \mathcal{N}(0, 1)$. Again, let us choose a sufficiently large T and fix some terminal condition such as $k_{T+1} = k^*$, so that the turnpike argument applies. Fair and Taylor (1983) proposed to construct a solution path to (S25) by setting all future innovations to their expected values, $\epsilon_1 = \epsilon_2 = \cdots = 0$. This produces a path on which technology evolves as $\ln(z_{t+1}) = \rho_t \ln(z_t)$ gradually converging to $z^* = 1$ and the model's variables gradually converge to the steady state. Note that only the first entry k_1 of the constructed path $(k_1, \ldots, k_T)$ is meaningful; the remaining entries $(k_2, \ldots, k_T)$ are obtained under a supplementary assumption of zero future innovations and they are only needed to accurately construct k_1. Thus, k_1 is stored and the rest of the sequence is discarded. By applying the same procedure to next state (k_1, z_1), we produce k_2, and so on until the path of desired length τ is constructed.

However, certainty equivalence approximation of Fair and Taylor (1983) has its limitations. It is exact for linear and linearized models, and it can be sufficiently accurate for models that are close to linear; see Gagnon and Taylor (1990) and Love (2010). However, it becomes highly inaccurate when either volatility and/or the degrees of nonlinearity increase; see our accuracy evaluations in the main text.

Another novelty of the EP method relative to shooting methods is that it iterates on the economy's path at once using Gauss–Jacobi iteration. This type of iteration is more stable than Gauss–Siedel and allows us to avoid explosive behavior. To be specific, it guesses the economy's path $(k_1, \ldots, k_{T+1})$, substitutes the quantities for $t = 1, \ldots, T+1$ in the right-hand side of T Euler equations (S25), respectively, and obtains a new path $(k_0, \ldots, k_T)$ in the left-hand side of (S25); and it iterates on the path until the convergence is achieved. Finally, Fair and Taylor (1983) proposed a simple procedure for determining T that ensures that a specific terminal condition used does not affect the quality of approximation, namely, they suggested to increase T (i.e., extend the path) until the solution in the initial period(s) becomes insensitive to further increases in T.

We now elaborate the description of the version of Fair and Taylor's (1983) method used to produce the results in the main text. We use a slightly different representation of the optimality conditions of the model (S5)–(S7) (we assume $\delta = 1$ and $u(c) = \ln(c)$ for expository convenience). The Euler equation and budget constraint, respectively, are

$$\frac{1}{c_t} = \beta E_t\left[\frac{1}{c_{t+1}}\big(1-\delta+z_{t+1}f'(k_{t+1})\big)\right],$$
$$c_t + k_{t+1} = (1-\delta)k_t + z_t f(k_t).$$

We combine the above two conditions to get

$$k_{t+1} = z_t f(k_t) - \left[E_t\left(\frac{\beta z_{t+1} f'(k_{t+1})}{z_{t+1}f(k_{t+1}) - k_{t+2}}\right)\right]^{-1}$$
$$\approx z_t f(k_t) - \frac{z^e_{t+1} f(k_{t+1}) - k_{t+2}}{\beta z^e_{t+1} f'(k_{t+1}))}, \tag{S26}$$

where the path for z^e_{t+1} is constructed under the certainty equivalence assumption that $\epsilon_{t+1} = 0$ for all $t \geq 0$. Under the conventional AR(1) process for productivity levels, this means that $\ln z^e_{t+1} = \rho \ln z^e_t$ for all $t \geq 0$, or equivalently $z^e_{t+1} = (z^e_t)^\rho$, where $z^e_0 = z_0$. To solve for the path of variables, we use derivative-free iteration in line with Gauss–Jacobi method as in Fair and Taylor (1983):

ALGORITHM 2 (Extended path (EP) framework by Fair and Taylor (1983)).

The goal of EP framework of Fair and Taylor (1983)

EFP is aimed at approximating a path for variables satisfying the model's equations during the first τ periods, that is, it finds $\widehat{k}_0, \dots, \widehat{k}_\tau$ such that $\|k_t - \widehat{k}_t\| < \varepsilon$ for $t = 1, \dots, \tau$, where $\varepsilon > 0$ is target accuracy, $\|\cdot\|$ is an absolute value, and k_t and $\widehat{k}_t$ are the t-period true capital stocks and their approximation, respectively.

Step 0: Initialization.

a. Fix $t = 0$ period state (k_0, z_0).

b. Choose time horizon $T \gg \tau$ and terminal condition $\widehat{k}_{T+1}$.

c. Construct and fix $\{z^e_{t+1}\}_{t=0,\dots,T}$ such that $z^e_{t+1} = (z^e_t)^\rho$ for all t, where $z^e_0 = z_0$.

d. Guess an equilibrium path $\{\widehat{k}^{(1)}_t\}_{t=1,\dots,T'}$ for iteration $j = 1$.

e. Write a t-period system of the optimality conditions in the form: $\widehat{k}_{t+1} = z^e_t f(\widehat{k}_t) - \frac{z^e_{t+1} f(\widehat{k}_{t+1}) - \widehat{k}_{t+2}}{\beta z^e_{t+1} f'(\widehat{k}_{t+1}))}$, where $\widehat{k}_0 = k_0$.

Step 1: Solving for a path using Gauss–Jacobi method.

a. Substitute a path $\{\widehat{k}^{(j)}_t\}_{t=1,\dots,T'}$ into the right-hand side of (S26) to find $\widehat{k}^{(j+1)}_{t+1} = z^e_t f(\widehat{k}^{(j)}_t) - \frac{z^e_{t+1} f(\widehat{k}^{(j)}_{t+1}) - \widehat{k}^{(j)}_{t+2}}{\beta z^e_{t+1} f'(\widehat{k}^{(j)}_{t+1}))}$, $t = 1, \dots, T$.

b. End iteration if the convergence is achieved $|\widehat{k}_{t+1}^{(j+1)} - \widehat{k}_{t+1}^{(j)}| <$ tolerance level. Otherwise, increase j by 1 and repeat Step 1.

The EP solution:

Use the first entry $\widehat{k}_1$ of the constructed path $\widehat{k}_1, \dots, \widehat{k}_T$ as an approximation to the true solution k_1 in period $t = 0$ and discard the remaining $k_2, \dots, k_T$ values.

In terms of our notations, Fair and Taylor (1983) used $\tau = 1$, that is, they kept only the first element $\widehat{k}_1$ from the constructed path $(\widehat{k}_1, \dots, \widehat{k}_T)$ and disregarded the rest of the path; then, they drew a next period shock z_1 and solved for a new path $(\widehat{k}_1, \dots, \widehat{k}_{T+1})$ starting from $\widehat{k}_1$ and ending in a given $\widehat{k}_{T+1}$ and stored $\widehat{k}_2$, again disregarding the rest of the path; and they advanced forward until the path of the given length τ is constructed. T is chosen so that its further extensions do not affect the solution in the initial period of the path. For instance, to find a solution $\widehat{k}_1$, Fair and Taylor (1983) solved the model several times under $T+1, T+2, T+3, \dots$ and checked that $\widehat{k}_1$ remains the same (up to a given degree of precision).

As is typical for fixed-point-iteration style methods, Gauss–Jacobi iteration may fail to converge. To deal with this issue, Fair and Taylor (1983) used damping, namely, they updated the path over iteration only by a small amount $k_{t+1}^{(j+1)} = \xi k_{t+1}^{(j+1)} + (1-\xi) k_{t+1}^{(j)}$ where $\xi \in (0, 1)$ is a small number close to zero (e.g., 0.01).

Steps 1a and *1b* of Fair and Taylor's (1983) method are called Type I and Type II iterations and are analogous to *Step 2* of the EFP method when the sequence of the decision functions is constructed. The extension of path is called Type III iteration and gives the name to Fair and Taylor (1983) method.

In our examples, we implement Fair and Taylor's (1983) method using a conventional Newton-style numerical solver instead of Gauss–Jacobi iteration; a similar implementation was used in Heer and Maußner (2010). The cost of Fair and Taylor's (1983) method can depend considerably on a specific solver used and can be very high (as we need to solve a system of equations with hundreds of unknowns numerically). In our simple examples, a Newton-style solver was sufficiently fast and reliable. In more complicated models, we are typically unable to derive closed-form laws of motion for the state variables, and derivative-free fixed-point iteration advocated in Fair and Taylor (1983) can be a better alternative.

Appendix D: Solving the test model using the associated stationary model

We first convert the nonstationary model (S5)–(S7) with labor-augmenting technological progress into a stationary model using the standard change of variables $\widehat{c}_t = c_t / A_t$ and $\widehat{k}_t = k_t / A_t$. This leads us to the following model:

$$\max_{\{\widehat{k}_{t+1}, \widehat{c}_t\}_{t=0,\dots,\infty}} E_0 \sum_{t=0}^{\infty} (\beta^*)^t \frac{\widehat{c}_t^{1-\eta}}{1-\eta} \tag{S27}$$

$$\text{s.t.} \quad \widehat{c}_t + \gamma_A \widehat{k}_{t+1} = (1-\delta)\widehat{k}_t + z_t \widehat{k}_t^{\alpha}, \tag{S28}$$

$$\ln z_{t+1} = \rho_t \ln z_t + \sigma_t \epsilon_{t+1}, \quad \epsilon_{t+1} \sim \mathcal{N}(0,1), \tag{S29}$$

where $\beta^* \equiv \beta\gamma_A^{1-\eta}$. We solve this stationary model by using the same version of the Smolyak method that is used within EFP to find a solution to T-period stationary economy.

After a solution to the stationary model (S27)–(S29) is constructed, a solution for nonstationary variables can be recovered by using an inverse transformation $c_t = \widehat{c}_t A_t$ and $k_t = \widehat{k}_t A_t$.

For the sake of our comparison, we also need to recover the path of nonstationary decision functions in terms of their parameters. Let us show how this can be done under polynomial approximation of decision functions. Let us assume that a capital policy function of the stationary model is approximated by complete polynomial of degree L, namely, $\hat{k}_{t+1} = \sum_{l=0}^{L}\sum_{m=0}^{l} b_{m+\frac{(l-1)(l+2)}{2}+1}\hat{k}_t^m z_t^{l-m}$, where b_i is a polynomial coefficient, $i = 0, \dots, L + \frac{(L-1)(L+2)}{2} + 1$. Given that the stationary and nonstationary solutions are related by $\hat{k}_{t+1} = k_{t+1}/(A_0\gamma_A^{t+1})$, we have

$$\begin{aligned} k_{t+1} &= A_0\gamma_A^{t+1}\hat{k}_{t+1} = A_0\gamma_A^{t+1}\sum_{l=0}^{L}\sum_{m=0}^{l} b_{m+\frac{(l-1)(l+2)}{2}+1}\hat{k}_t^m z_t^{l-m} \\ &= A_0\sum_{l=0}^{L}\sum_{m=0}^{l} \gamma_A{}^{1-(m-1)t} b_{m+\frac{(l-1)(l+2)}{2}+1} k_t^m z_t^{l-m}. \end{aligned} \tag{S30}$$

For example, for first-degree polynomial $L = 1$, we construct the coefficients vector of the nonstationary model by premultiplying the coefficient vector $b \equiv (b_0, b_1, b_2)$ of the stationary model by a vector $(A_0\gamma_A^{t+1}, A_0\gamma_A, A_0\gamma_A^{t+1})^\top$, which yields $b_{t+1} \equiv (b_0 A_0\gamma_A^{t+1}, b_1 A_0\gamma_A, b_2 A_0\gamma_A^{t+1})$, $t = 0, \dots, T$, where T is time horizon (length of simulation in the solution procedure). Note that a similar relation will hold even if the growth rate γ_A is time variable.

Appendix E: Sensitivity results for the model with labor-augmenting technological progress

In this appendix, we provide sensitivity results for the model with labor-augmenting technological progress. Table 2 contains the results on accuracy and cost of the version of the EFP method studied in Section 5. We use $\tau = 200$ and $T = 400$ and consider several alternative parameterizations for $\{\eta, \sigma_\epsilon, \gamma_A\}$.

Figure S2 plots a maximum unit-free absolute difference between the exact solution for capital and the solution delivered by the EFP at $\tau = 100$. The difference between the solutions is computed across 1000 simulations. We use $T = \{200, 300, 400, 500\}$, $\eta = \{1/3, 1, 3\}$, and decision rules produced by the T-period stationary economy and zero capital assumption as terminal conditions.

TABLE 2. Sensitivity analysis for the EFP method.

Parameters	Model 1	Model 2	Model 3	Model 4	Model 5	Models 6	Model 7
η	5	5	5	5	0.1	1	10
σ_ϵ	0.03	0.03	0.03	0.01	0.01	0.01	0.01
γ_A	1.01	1.00	1.05	1.01	1.01	1.01	1.01
Mean errors across t periods in $\log_{10}$ units							
$t \in [0, 50]$	−7.01	−6.67	−7.34	−7.03	−7.03	−6.61	−7.30
$t \in [0, 100]$	−6.82	−6.44	−7.25	−6.84	6.92	−6.48	−7.08
$t \in [0, 150]$	−6.73	−6.33	−7.22	−6.76	−6.89	−6.43	−6.98
$t \in [0, 175]$	−6.70	−6.29	−7.22	−6.74	−6.87	−6.41	−6.95
$t \in [0, 200]$	−6.68	−6.26	−7.21	−6.72	−6.87	−6.37	−6.93
Maximum errors across t periods in $\log_{10}$ units							
$t \in [0, 50]$	−6.42	−6.31	−7.13	−6.66	−6.08	−6.24	−6.81
$t \in [0, 100]$	−5.99	−6.12	−7.05	−6.54	−5.97	−6.18	−6.36
$t \in [0, 150]$	−5.98	−6.04	−7.05	−6.52	−5.97	−6.18	−6.35
$t \in [0, 175]$	−5.98	−6.01	−7.05	−6.52	−5.97	−6.13	−6.33
$t \in [0, 200]$	−5.92	−5.99	−7.05	−6.51	−5.96	−5.88	−6.24
Running time, in seconds							
Solution	225.9	150.0	193.0	216.98	836.5	300.7	245.9
Simulation	5.6	5.7	5.8	5.66	5.6	5.6	5.7
Total	231.6	155.7	198.8	222.64	842.1	306.3	251.6

Note: "Mean errors" and "Maximum errors" are, respectively, mean and maximum unit-free absolute difference between the exact solution for capital and the solution delivered by EFP under the parameterization in the column. The difference between the solutions is computed across 100 simulations. The time horizon is $T = 400$, and the terminal condition is constructed by using the T-period stationary economy in all experiments.

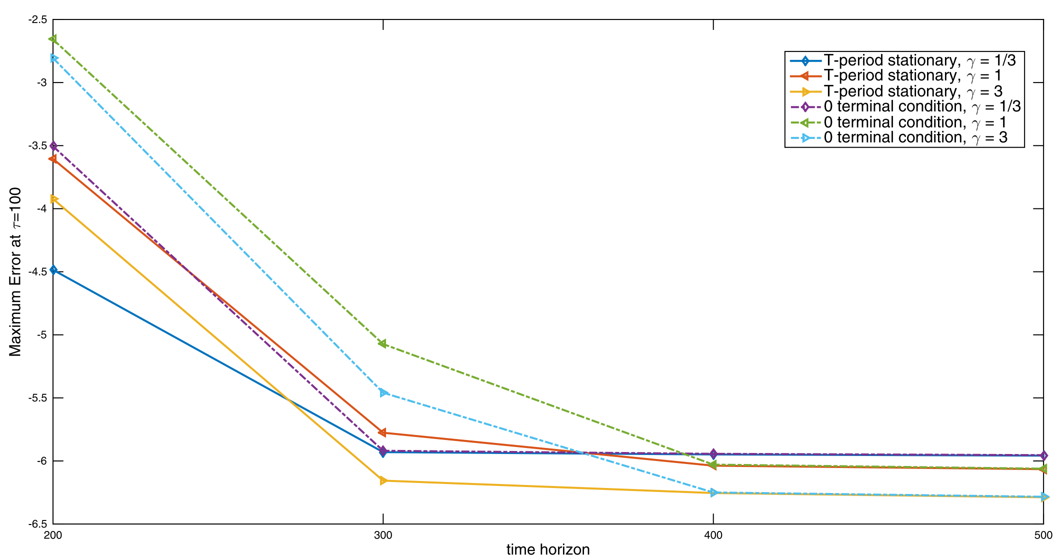

FIGURE S2. Sensitivity analysis for the EFP method.

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